METU-MATHEMATICS DEPARTMENT
Graduate Preliminary Examinations

Geometry

Duration: 3 hours

February 18, 2005

1. Consider the set $M = \{(x, y, z, w) \in \mathbb{R}^4 \mid x^2 + y^2 = 1 \ , \ z^2 + w^2 = 1\} \subseteq \mathbb{R}^4$.

(**a**) Prove that M is an (imbedded) submanifold of $\mathbb{R}^4$.

(**b**) Describe the tangent vectors of M at an arbitrary point $(a, b, c, d) \in M$.

(**c**) Write down a nowhere vanishing vector field on M .

(**d**) Let $\omega = (ydx - xdy) \wedge (wdz - zdw) \in \Omega(\mathbb{R}^4)$. Show that $\int_M i_\star(w) > 0$ where $i : M \to \mathbb{R}^4$ is the inclusion map (Hint: Write a local parametrization for M).

(**e**) A consequence of Poincaré Lemma is that every closed form on $\mathbb{R}^n$ for any n is also exact. Prove that there exists no 4-form $\theta \in \Omega(\mathbb{R}^4)$ with $d\theta = 0$ such that $\int_M i^\star(\theta) \neq 0$.

2. Consider the the $(k-1)$ dimensional sphere S^{k-1} as a submanifold of S^k via the usual embeddding $(x_1, x_2, \ldots, x_k) \to (x_1, x_2, \ldots, x_k, 0)$. Show that the orthogonal complement to $T_p(S^{k-1})$ in $T_p(S^k)$ is spanned by the vector $(0, 0, \ldots, 1)$.

3. Let ω be a compactly supported 2-form

$w = f_1 \ dx_2 \wedge dx_3 + f_2 \ dx_3 \wedge dx_1 + f_3 \ dx_1 \wedge dx_2$

on $\mathbb{R}^3$. Let S be the graph of a function $G : \mathbb{R}^2 \to \mathbb{R}$. Compute the integral $\int_S \omega$, and show that it is equal to $\int_{\mathbb{R}^2} (\vec{F}.\vec{u}) \ \|\vec{n}\| \ dx_1 \wedge dx_2$ where $\vec{F} = (f_1, f_2, f_3)$, $\vec{u} = \frac{\vec{n}}{\|\vec{n}\|}$ with $\vec{n} = (-\frac{\partial G}{\partial x_1}, -\frac{\partial G}{\partial x_2}, 1)$.

4. Consider the sets

$M_1 = \{[u, v, w] \in \mathbb{R}P^2 \mid u^2 + v^2 = w^2\} \subseteq \mathbb{R}P^2$.

$M_2 = \{[u, v, w] \in \mathbb{R}P^2 \mid u^2 - v^2 = w^2\} \subseteq \mathbb{R}P^2$.

(**a**) Prove that M_1 is an (imbedded) submanifold of $\mathbb{R}P^2$ diffeomorphic to $\mathbf{S}^1$ (Hint: Consider the image of M_1 under a suitable chart of $\mathbb{R}P^2$).

(**b**) Find a diffeomorphism $F : \mathbb{R}P^2 \to \mathbb{R}P^2$ such that $F(M_1) = M_2$.