Geometry with a twist: Model Theory of Fields at MSRI

Carol Wood

In his recent book on model theory, Wilfrid Hodges describes a current view of model theory via the slogan, "model theory is algebraic geometry minus fields". What then is meant by the model theory of fields? Is model theory of fields equal to (or posing as) algebraic geometry?

Well, no. In dry terms, model theory is that branch of logic concerned principally with semantics; its fundamental arena of study is the class of abstract structures and definable sets within these structures, and its most typical language is that of first order logic. The abstraction serves as a lens through which one can understand various parts of mathematics. To make appropriate connections involves finding the right bits of mathematics on which to focus, and also requires selecting the proper level of magnification. The slogan proposed by Lou van den Dries in his introductory lectures at MSRI is that model theory is the "geography of tame mathematics". The lens analogy would be that a mathematical topic is tame provided the correct direction is chosen. The boundaries of tameness are found by setting the level of magnification appropriately to the current state of knowledge.

It is my impression, based on casual conversations, that nonlogicians think of logic in terms of set theory. This is what they mention, understandably so, due to the spectacular and fundamental work of 30+ years ago involving the axiom of choice and the continuum hypothesis. Senior mathematicians may recall taking notice of these developments at the time, looking up long enough to determine whether these results had any impact on their work. Most mathematicians have never experienced a similar "heads up" from model theory, and never will. If the first proof of a theorem came from model theory, a reasonable reaction was to seek a proof cast in terms of the existing tools of the subject. There have been exceptions in the past, e.g., in parts of nonstandard analysis, and in Ax's beautiful work on finite fields.

Recent developments in the model theory of fields suggest that the effort required in excising the model theory wherever possible may be greater than learning the model theory involved. The translation difficulties arise because model theory has gained technical power and sophistication over the past thirty years, making it harder to remove from the picture, as well as harder to explain to the non-expert consumer. This is a trend in mathematics in the latter part of this century:

there are few safe corners in which one can comfortably work without risk of needing to know about seemingly unrelated areas.

Likewise, model theorists find themselves in need of an array of mathematics outside logic, such as Galois theory, etale cohomology, germs of analytic functions, intersection theory and group representations.

A central goal of the Model Theory of Fields program at MSRI is to facilitate interaction between model theory and two broad areas of mathematics: (i) number theory and algebraic geometry, and (ii) real algebraic geometry and rigid analytic geometry.

The introductory workshop in January gave us a great start, thanks to the organizers Haskell, Pillay, and Steinhorn, with overarching themes of dimension theory, geometry, and model theory of fields. This workshop was attended by more than the usual (i.e. model theorist) suspects, no doubt due in part to the talks by well known mathematicians outside model theory, including Edward Bierstone and Barry Mazur.

During the course of the semester, the added learning beyond a typical research semester provides a very rich, if daunting, educational task. The post-doctoral fellows play an important, perhaps unusual role, in our program. Their expertise and familiarity with recent applications makes them able teachers of their more senior colleagues. Participation by senior researchers outside model theory has also been substantial and essential; Voloch serves as organizer as well. Additional and fruitful interaction occured when Pillay and Scanlon gave expository talks on model theory and its applications in arithmetic geometry during a March workshop on Arithmetic Geometry at the Newton Institute.

In describing the perspective taken in the model theory of fields, a nearly irresistible starting place is with algebraically closed fields. Algebraically closed fields provide a natural setting, indeed provably the natural setting, in which to study polynomial equations over fields. Algebraically closed fields are tame, differing from one another model-theoretically only in characteristic. Algebraic geometry carries the day. One has a good notion of dimension, plus elegant descriptions of definable sets of any dimension, as boolean combinations of solution sets to polynomial equations. Definable subsets of algebraically closed fields are easily described: finite sets and the complements of finite sets.

This is in stark contrast to the absence of geometry one finds in studying equations over the rationals. Julia Robinson showed that definable sets in $\mathbb{Q}$ can be arbitrarily complicated. In $\mathbb{Q}$, pathologies of undecidability make the terrain very far from tame.

For some time model theorists have considered intermediate cases. Hints of good behavior, in combination with existing mathematics, led to manageable theories, sometimes set in enriched field languages (hence the "twist" of the title). The new symbols of the language may extend the family of definable sets and may restrict the class of fields; one or the other must occur, but not necessarily both. Two examples should serve to illustrate this phenomenon: the real numbers $\mathbb{R}$ and $\mathbb{Q}(x_0, x_1, \dots)^{alg}$, the algebraic closure of the function field in $x_0, x_1, \dots$ over the rationals.

Each is a structure for the language of rings, with addition and multiplication as basic notions, and with symbols for 0 and 1 (mostly for convenience). First consider $\mathbb{R}$. Artin and Schreier captured the algebraic properties of $\mathbb{R}$ in the notion of a real closed field. Tarski showed that the axioms for real closed fields give a complete first order description of the reals as fields. (This is itself a useful fact but not one on which we dwell just now; rather, we want to stake out larger tame territory.) The definable sets are best described using an auxiliary symbol for order $<$. This does not change the class of definable sets in real closed fields: $a < b$ just in case $a - b$ is a non-zero square. The definable subsets of $\mathbb{R}^n$ then are exactly the semi-algebraic sets, i.e., given by boolean combinations of conditions of the form $f(x_1, \dots, x_n) = 0$ and $g(x_1, \dots, x_n) > 0$, where f and g are polynomials over $\mathbb{R}$. This collection is closed under projections, and thus more complicated formulas are equivalent to the boolean combinations above. Each definable subset of $\mathbb{R}$ itself is a finite union of points and intervals. For subsets of $\mathbb{R}^n$ a cell decomposition is available, and all definable sets have finitely many connected components.

Now what happens if one enriches the language in a less innocent way? With a bad choice, such as the sine function, one goes outside the boundaries of tame; the set of zeros of the sine function codes pathologies, as in the case of the rationals. Consider instead adding the exponential function e^x to the field of real numbers. The collection of definable subsets of $\mathbb{R}^2$ has been extended, a fact easily seen by comparing the growth rate of field definable (=polynomial) functions with that of e^x. In 1991, Wilkie showed that adding e^x results in a well-behaved structure, in which the definable sets are again built up from finite pieces. His work opened up investigations of ever richer classes of definable sets in tame territory. The results are of use in real geometry and in our understanding of asymptotic behavior.

The second field $K = \mathbb{Q}(X)^{alg}$ is, qua field, algebraically closed of characteristic zero; as with the first example, nothing further can be expressed about K in the language of fields. Definable subsets of K are either finite or cofinite. By adding a derivation δ to the language, setting $\delta(x_0) = 1$, we can extend - in many ways - the derivation to all of K to obtain a structure which is the differential analogue of algebraically closed, called differentially closed by its inventor, Abraham Robinson. We can also arrange that $\mathbb{Q}^{alg}$ is the field of δ-constants of K. The collection of definable sets has been enriched- e.g.,$\mathbb{Q}^{alg}$ is definable now- and yet certain good model theoretical behavior is retained. Thus in the differential setting model theory provides useful notions of dimension, and a rich supply of 1-dimensional sets with which to capture and describe certain diophantine sets, as has been done in striking ways by Hrushovski.

The above expansions, and others in current use, have been known for some time. What is new is the application of the latest tools of model theory, with new geometries associated to certain definable families of sets. Ideas of Morley and Vaught were developed to considerable depth; Shelah pushed the area of stability theory very far technically, as he worked toward his goal of classifying theories in general. (A first cut for tameness is the notion of stable.) In the case of fields, if one has chosen the language well, guided by an understanding of the field's internal mathematics, then stability theory leads to a geometry of definable sets. With a bit of luck, this geometry has a well behaved dimension theory, thus making available the most sophisticated tools of model theory. Often one can move from results about dimension 1 to results about arbitrary dimension. These higher dimensional consequences are sometimes obscure when viewed outside the model theoretical framework. Difficulties do arise in understanding the underlying mathematics of the 1-dimensional sets, and even of sets which are infinite intersections of definable sets, and not themselves definable. But model theory kicks in, employing the analysis of stable theories and related phenomena, and applying as appropriate *o*-minimality, various notions of rank, of Zariski geometries, of 1-based groups,etc.. This analysis of fields has fed back into the abstract model theory as well. One sees this in the renewed interest -and exciting results by Kim- in Shelah's notion of simple theories, a broader class than stable.

I have attempted here only to suggest something of the manner in which model theorists approach mathematics, and of the importance to the present program of interactions among research areas. Good expositions of the main areas of our program can be found via the MSRI web site, together with the introductory workshop lectures and the program for our June workshop. Recent articles by Marker in the AMS Notices and by Pillay in the AMS Bulletin provide excellent accounts of topics central to our program.

Note: I thank Concha Gomez for pointing out the paradox when one combines Hodge's slogan and our program's name!