An Analytic Application of Zorn's Lemma

İlksen Acunalp

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Hahn-Banach Theorem: Let V be a vector space over the field $\mathbf{R}$ and $p : V \to \mathbf{R}$ be such that $p(x+y) \leq p(x) + p(y)$ and $p(ax) = ap(x)$ for all $x, y \in V$ and $a \geq 0$. If f is a linear functional on a linear subspace S of V such that $f(s) \leq p(s)$ for all $s \in S$, then there exists a linear functional F on V such that F extends f and $F(x) \leq p(x)$ for all $x \in V$.

Proof: Step 1: Set $S = \{(W, f) : W \leq V$ is a subspace, $f : W \to \mathbf{R}$ is linear, $f(w) \leq p(w)$ for all $w \in W\}$. Since $\{0\} \in S$, $S \neq \emptyset$.

Step 2: Define a partial order on S as follows:

$$(W, f) \leq (U, g) \Leftrightarrow W \leq U \text{ and } g|_W = f.$$

$\leq$ is reflexive, because $(W, f) \leq (W, f)$ since $W \leq W$ and $f|_W = f$.

$\leq$ is transitive, since if $(W, f) \leq (f, g)$ and $(U, g) \leq (Y, h)$ then $W \leq U$ and $g|_W = f$, $U \leq Y$ and $h|_U = z$. So $W \leq Y$ and $h|_W = f$, thus $(W, f) \leq (Y, h)$.

$\leq$ is anti-symmetric, since $(W, f) \leq (U, g)$ and $(U, g) \leq (W, f)$ imply $W \leq U$ and $U \leq W$ and hence $U = W$. Also $f = g$.

Step 3: Show that every chain in S has an upper bound in S.

Take an arbitrary chain $\{(W_i, f_i) | i \in I\}$ in S, then consider $(\cup W_i, f)$, where $f : \cup W_i \to \mathbf{R}$, defined as $f(w) = f_j(w)$ if $w \in W_j$.

Since $W_i \leq \cup W_i$ and $f|_{W_i} = f_i$, $(\cup W_i, f)$ is an upper bound.

Note $(\cup W_i, f) \in S$ because $\cup W_i$ is a subspace of V and f is linear.

Step 4: By **Zorn's Lemma**, S has a maximal element (T, F).

Step 5: Prove $T = V$.

To obtain a contradiction assume that $V \neq T$ and $w \in V \setminus T$. We will show that this contradicts the maximality of F by constructing a $G : V_0 = span(T \cup \{w\}) \to V_0$ extending F.

First notice that for every $s, t \in T$,
$F(s) + F(t) = F(s + t)$
$\leq p(s + t) = p(s - w + t + w)$
$\leq p(s - w) + p(t + w)$ and so

$$-p(s - w) + F(s) \leq p(t + w) - F(t).$$

In particular,

$$\sup_{t \in T}[-p(t - w) + F(t)] \leq \inf_{t \in T}[p(t + w) - F(t)].$$

Choose α such that

$$\sup[-p(t - w) + F(t)] \leq \alpha \leq \inf[p(t + w) - F(t)].$$

Then
$\alpha \leq \inf[p(t + w) - F(t)]$

$$\text{and } -\alpha \leq \inf[p(t - w) - F(t)]. \qquad (\star)$$

Now for $a \in \mathbf{R}$ and $t \in T$ define

$$G(aw+t) = a\alpha + F(t).$$

G is a functional on V_0 that extends F. We have to show only that $G(s) \leq p(s)$ for all $s \in V_0$. So let $s = aw + t$ with $t \in T$, $a \in \mathbf{R}$.

If $a = 0$ then $G(aw+t) = F(t) \leq p(t) = p(aw+t)$.

If $a > 0$ then by the first part of $(\star)$.

$$\begin{aligned} G(aw+t) &= a\alpha + F(t) \\ &= a[\alpha + F(\frac{t}{a})] \\ &\leq a[p(\frac{t}{a} + w) - F(\frac{t}{a}) + F(\frac{t}{a})] \\ &= ap(\frac{t}{a} + w) \\ &= p(t + aw) \end{aligned}$$

If $a = -b < 0$, by the second part of $(\star)$.

$$\begin{aligned} G(aw+t) &= -b\alpha + F(t) \\ &= b[-\alpha + F(\frac{t}{b})] \\ &\leq b[p(\frac{t}{b} - w) - F(\frac{t}{b}) + F(\frac{t}{b})] \\ &= bp(\frac{t}{b} - w) \\ &= p(t + w) \end{aligned}$$

This finishes the proof. *QED*