An Algebraic Application of Zorn's Lemma

Can Deha Karıksız

12 May 2004

Definitions: An abelian group $(V, +)$ is said to be a **vector space** over a field $\mathbf{F}$, if there is an operation $\mathbf{F} \times \mathbf{V} \to \mathbf{V}$ $(k, v) \to kv$ such that
(I) $(k + \ell)v = kv + \ell v$
(II) $k(v + w) = kv + kw$
(III) $k(\ell v) = (k\ell)v$
(IV) $0v = 0$
(V) $1v = v$ $\forall k, \ell \in \mathbf{F}$ and $v, w \in V$.

A subset S of a vector space V over a field $\mathbf{F}$ is a **subspace** of V if it is a linear space when considered with the same operations.

Examples: $\mathbf{R^n}$ is a vector space over $\mathbf{R}$, $\mathbf{R}$ is a vector space over $\mathbf{Q}$, the set of all polynomials with real coefficients is a vector space over $\mathbf{R}$.

A subset S of a vector space V over a field $\mathbf{F}$ is **linearly independent** if for every finite number of distinct elements $v_1, \cdots, v_n \in S$ and every $k_1, \cdots, k_n \in \mathbf{F}$, the condition $k_1v_1 + \cdots + k_nv_n = 0$ implies $k_1 = \cdots = k_n = 0$.

S **spans** V if every $v \in V$ can be represented as $v = k_1v_1 + \cdots + k_nv_n$ for some $v_1, \cdots, v_n \in S$ and $k_1, \cdots, k_n \in \mathbf{F}$.

A **basis** of V is a linearly independent subset of V that spans V.

Proposition: If β is a basis of V, then every $v \in V$ has a unique representation $v = k_1v_1 + \cdots + k_nv_n$ where $v_1, \cdots, v_n \in \beta$ are distinct and $k_1, \cdots, k_n \in \mathbf{F}$.

Proof. Let $v \in V$. Let $v = k_1v_1 + \cdots + k_nv_n$ and $v = \ell_1v_1 + \cdots + \ell_nv_n$ be different representations of v, then $(\ell_1 - k_1)v_1 + \cdots + (\ell_n - k_n)v_n = v - v = 0$.
Since β is a basis, β is linearly independent. So, $\ell_1 - k_1 = \cdots = \ell_n - k_n = 0$.
Thus, $\ell_i = k_i$ for $i = 1, \cdots, n$. QED

Theorem: If S_0 is a linearly independent subset of a vector space V over $\mathbf{F}$, then there exists a basis β of V that contains S_0. In particular, every vector space has a basis.

Proof: The additional part follows from the main part, since the empty set is linearly independent in any vector space. (Set $S_0 = \phi$.)
Let $\mathcal{F} = \{S \subset V : S_0 \subseteq S$ and S is linearly independent in $V\}$.
Claim: $\mathcal{F}$ satisfies the assumptions of Zorn's Lemma.

Zorn's Lemma: Suppose $\mathcal{F}$ is nonempty partially ordered set and every chain $\mathcal{G}$ of elements of $\mathcal{F}$ has an upper bound in $\mathcal{F}$ (that is, there is an element $M \in \mathcal{F}$ with $U \subseteq M$ for all $U \in \mathcal{G}$). Then $\mathcal{F}$ has a maximal element.

Proof of the claim:

Note that $\mathcal{F}$ is nonempty since $S_0 \in \mathcal{F}$, and $\mathcal{F}$ is partially ordered by inclusion.

Let $\mathcal{G} \subset \mathcal{F}$ be a chain in $\mathcal{F}$. It is clear that $\cup\mathcal{G}$ is an upper bound for $\mathcal{G}$. Now let's show that $\cup\mathcal{G} \in \mathcal{F}$.

Let $v_1, \cdots, v_n$ be different elements of $\cup\mathcal{G}$ and choose $k_1, \cdots, k_n \in \mathbf{F}$ such that $k_1v_1 + \cdots + k_nv_n = 0$.

For every $i \in \{1, \cdots, n\}$ choose $G_i \in \mathcal{G}$ such that $v_i \in G_i$.

Since $\{G_1, \cdots, G_n\}$ is a finite subset of a linearly (totally) ordered set $\mathcal{G}$, we can find the largest element, say G_j, in this set.

Then $v_i \in G_i \subset G_j$ for all $i \in \{1, \cdots, n\}$. Hence all v_i's are in a linearly independent set G_j. Thus $k_1v_1 + \cdots k_nv_n = 0$ implies $k_1 = \cdots = k_n = 0$.

So, we can apply **Zorn's Lemma**.

Let β be a maximal element in $\mathcal{F}$. Next we'll show that β is a basis for V.

- β is linearly independent, since β belongs to $\mathcal{F}$.
- Does β span V?

To obtain a contradiction, assume that there is a $v \in V$ such that $v \neq k_1v_1 + \cdots + k_nv_n(\star)$ for every $v_1, \cdots, v_n \in V$ and $k_1, \cdots, k_n \in F$.

We'll show that this implies that $\beta \cup \{v\}$ is linearly independent in V, which contradicts the maximality of β in $\mathcal{F}$.

Choose different elements $v_1, \cdots, v_n$ from β and $k_0, \cdots, k_n \in F$ such that $k_0v + k_1v_1 + \cdots + k_nv_n = 0$. There are two cases:

Case I: $k_0 = 0$ then $k_1v_1 + \cdots + k_nv_n = 0$ hence $k_1 = \cdots = k_n = 0$ since $v_1, \cdots, v_n \in \beta$

Case II: $k_0 \neq 0$ then $k_0v = -k_1v_1 + \cdots - k_nv_n$

$$v = (-\frac{k_1}{k_0})v_1 + \cdots - (\frac{k_n}{k_0})v_n$$

This contradicts $(\star)$.

In conclusion, we have $k_0v + k_1v_1 + \cdots + k_nv_n = 0$ implies $k_0 = \cdots = k_n = 0$.

So, $\beta \cup \{v\}$ is linearly independent in V, which contradicts the maximality of β in $\mathcal{F}$. Thus, β spans V.

β is linearly independent and β spans V together imply β is a basis for V. QED