Banach-Tarski Paradox

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PART A

Theorem 1: S^1 is countably SO_2-paradoxical.

Proof: Consider the RSO_2, the subgroup of SO_2 generated rotations of rational multiples of 2π radions. Let H be a **choice set** for cosets of SO_2/RSO_2. Now let $M = \{\sigma(1,0) : \sigma \in H\}$.

Since RSO_2 is countable, we may enumerate it by p_i. Let $M_i = p_i(M)$. Then $\{M_i\}$ is countable partition of S^1 and moreover, all the M_i are congruent to each other by rotation.

Hence each set in $\{M_2, M_4, M_6, \cdots\}$ may be individually rotated to yield $\{M_1, M_2, M_3, \cdots\}$ whose union is S^1. We can clearly do the some for $\{M_i : i \text{ odd }\}$. Thus S^1 is countably SO_2-paradoxical. QED

Theorem 2: $S^1 \setminus \{pt\}$ is equidecomposable to S^1.

Proof: Consider $\mathbf{R^2}$ identified with $\mathbf{C}$. L et the pt be $1 = e^{i0}$. Let $A = \{e^{in} : n \in \{1, 2, 3, \cdots\}\}$, and let $B = (S^1 \setminus \{1\}) \setminus A$ be everything else. The points e^{in} are unique, since 2π is irrational. Then, leaving b fixed, rotate A by 1 radian. We obtain a complete circle. QED

PART B

Definition. A group G is called *paradoxical* if it is G-paradoxical.

Theorem 1: The free group F on two generators σ, τ is F-paradoxical.

Proof: Let $F = \{1\} \cup B(\sigma) \cup B(\sigma^{-1}) \cup B(\tau) \cup B(\tau^{-1})$ where all sets are pairwise dijoint. But $F = B(\sigma) \cup \sigma B(\sigma^{-1})$ and $F = B(\tau) \cup \tau B(\tau^{-1})$, which shows that F is paradoxical. QED

Theorem 2: Suppose G is a paradoxical group acting on X without any nontrivial fixed points. Then X is G-paradoxical.

Proof: Let $A_i, B_j \subseteq G$ and $g_i, h_j \in G$ be the sets and transformations "witnessing" that G is paradoxical. Take a choice set M from the G-orbits in X. Now $\{g(M) : g \in G\}$ partitions X because G acts without nontrivial fixed points. Let $A'_i = \cup\{g(M) : g \in A_i\}$ and $B'_j = \cup\{g(M) : g \in B_j\}$. Then $\{A'_i\}$ and $\{B'_j\}$ are all pairwise disjoint. and we can see that $X = \cup\{A'_i\} = \cup\{B'_j\}$. QED

Theorem 3: If a group G contains a paradoxical subgroup H, then G is paradoxical.

Proof:H acts on G by left multiplication, without nontrivial fixed points. (Inverses prevent this from happening). So G is H-paradoxical by Theorem 2. But then G is G-paradoxical. QED

Corollary: Any group with a free subgroup of rank 2 is paradoxical.

PART C

Theorem 1: There exist two independent rotations ϕ, ρ in $\mathbf{R}^3$ which fix the origin. Hence SO_n is paradoxical for all $n \geq 3$.

Proof (Outline): Most pairs of rotations are independent, i.e. $arccos(r)$ where r is any rational $\neq 0$, $\pm(1/2)$, ± 1. Take the rotations by arccos(3/5) around the x-axis and the z-axis.

Theorem 2 (Hausdorff Paradox): There is a countable set D such that S^2/D is SO_3-paradoxical.

Proof: Let F be the group generated by the rotations constructed above and D be the points on S^2 that are fixed by some non-identity element of F. By Theorem 2 (Part B), $S^2 \setminus D$ is SO_3-paradoxical. QED

Theorem 3: S^2 and $S^2 \setminus D$ are SO_3 equidecomposable.

Proof: Similar to the proof of theorem 2 (Part a).

PART D

Theorem 1 (Banach - Tarski Paradox): B^3, the solid ball in R^3, is G_3-paradoxical.

Proof: Since S^3 is paradoxical, we obtain a paradox for any "thickened" shell by producing a paradox for one sphere in this shell, and then putting points along a radius in the same piece in a decomposition. In particular, we see that $B^3 \setminus \{0\}$ is paradoxical.

Choose a broken circle $C' \subset B^3\{0\}$ with the origin as its broken point.

$$\begin{aligned}
B^3 \setminus \{0\} &= B^3 \setminus (\{0\} \cup C') \cup C' \\
&\sim B^3 \setminus (\{0\} \cup C') \cup (C' \cup \{0\}) \\
&= B^3 \\
B^3 &= B^3 \setminus \{0\} \cup \{0\} \\
&\sim B^3 \setminus \{0\} \cup B^3 \setminus \{0\} \cup \{0\} \\
&= B^3 \cup B^3 \\
&\sim B^3 \cup B^3
\end{aligned}$$

QED