M E T U
Department of Mathematics

	Introduction to Ordinary Differential Equations Second Midterm	
Code : *Math 253* Acad. Year : *2004-2005* Semester : *Fall* Instructor : Date : *27.11.2004* Time : *14.00* Duration : *70 minutes*	Last Name: Name : Student No: Department Section Signature : **3** Questions on **4** Pages Total **35** Points	

1	2	3		

(12 pts.) Q1. Given that f(x) = cosx is a solution of $\left(D^4 + D^3 - D^2 + D - 2\right) y = 0$,
find the general solution of $\left(D^4 + D^3 - D^2 + D - 2\right) y = 2x + 1$.

Answer:
Since $cos(x)$ is a solution, $\pm i$ is a root of $\left(D^4 + D^3 - D^2 + D - 2\right)$.

So,$(D+i)(D-i) = D^2 + 1$ divides $\left(D^4 + D^3 - D^2 + D - 2\right)$.

By dividing, we get the factorization
$\left(D^4 + D^3 - D^2 + D - 2\right) = (D^2+1)(D^2+D-2) = (D^2+1)(D+2)(D-1).$

Thus,$y_c = c_1 \cos x + c_2 \sin x + c_3 e^{-2x} + c_4 e^x.$

Since $g(D) = D^2$ is the smallest order such that $D^2(x+1) = 0$ and the roots of $D^2 = 0$,only 0 which is different than $\pm 1, -2, 1,$
$y_p = A + Bx$ by the method of undetermined coefficients.

Therefore,$\left(D^4 + D^3 - D^2 + D - 2\right)(A + Bx) = D(A + Bx) - 2(A + Bx) = B - 2A - 2Bx = 2x + 1.$

This gives $B = -1, A = -1$.

Then, $y_p = -1 - x.$

Thus,$y = y_c + y_p = c_1 \cos x + c_2 \sin x + c_3 e^{-2x} + c_4 e^x - 1 - x$

is the general solution.

(11 pts.) Q2. Find the general solution of $y'' - 2y' + y = e^x x^{-1}$,where $0 < x < \infty$.

Answer:
$L(D) = D^2 - 2D + 1 = (D-1)^2$ has only the real root 1.

Then, $y_c = c_1' e^x + c_2' x e^x$.

Using variation of parameters, we have $y_p = Ae^x + Bxe^x$ with the equations

(1) $A'e^x + B'xe^x = 0$ and

(2) $A'e^x + B'(e^x + xe^x) = e^x x^{-1}$

Then ,(1) and (2) give $B'e^x = e^x x^{-1}$.

Then, $B' = x^{-1}$ and from (1) , $A' = -1$

Then,by integrating, $B = \ln x$ and $A = -x$

So,$y_p = -xe^x + xe^x \ln(x)$

Therefore, $y = y_c + y_p = c_1' e^x + c_2' x e^x - xe^x + xe^x \ln x$

,or $y = c_1' e^x + c_2' x e^x + xe^x \ln x$ is the general solution.

(12pts)Q3. Find the general solution of the equation $y'' - 2xy' - 4y = 0$ about the point $x_0 = 0$.

Find also the radius of convergence of the general solution.

Answer=

Since $x_0 = 0$ is a ordinary point and the equation has no singular point,

there exists a valid solution in the form $y = \sum_{n=0}^{\infty} c_n x^n$,valid on $|x| < \infty$.

Then, $y = \sum_{n=0}^{\infty} c_n x^n,\ y' = \sum_{n=1}^{\infty} nc_n x^{n-1},\ y'' = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}$

Then ,from the equation, we have

$$\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n - 2\sum_{n=0}^{\infty} nc_n x^n - 4\sum_{n=0}^{\infty} c_n x^n = 0$$

Then,

$$\sum_{n=0}^{\infty}[(n+2)(n+1)c_{n+2} - 2\ nc_n - 4c_n]x^n = 0$$

Hence, we get a recurrence relation $(n+2)(n+1)c_{n+2} - 2(n+2)c_n = 0$ for $n = 0, 1, 2...$

$(n+2) \neq 0$ implies that $c_{n+2} = \frac{2}{n+1}c_n$ for $n = 0, 1, 2...$

Then,

$n = 0,$	$c_2 = \frac{2}{1}c_0$	$; n = 1$	$c_3 = \frac{2}{2}c_1$
$n = 2,$	$c_4 = \frac{2}{3}c_2$	$; n = 3$	$c_5 = \frac{2}{4}c_3$
$n = 4,$	$c_6 = \frac{2}{5}c_4$	$; n = 5$	$c_7 = \frac{2}{6}c_5$

...

From these, we can see that

$c_{2n} = \frac{2}{2n-1}c_{2n-2} = \frac{2^n}{1.3.5...2n-1}c_0$

and

$c_{2n+1} = \frac{2}{2n}c_{2n-1} = \frac{1}{1.2.3...n}c_1$

Therefore,

$y = c_0[1 + \sum_{n=1}^{\infty} \frac{2^n}{1.3.5...(2n-1)}x^{2n}] + c_1[x + \sum_{n=1}^{\infty} \frac{1}{n!}x^{2n+1}]$ is the general solution.