MATH 253

Exam I Solutions:

(10 pts.)Q1. Consider the initial value problem:
$2xyy' = 4x^2 + 3y^2, y(1) = 1.$

(2 pts.)a)Explain why the initial value problem has a unique solution:

$f(x, y) = y' = \frac{4x^2+3y^2}{2xy}$.Then,$\frac{\partial f(x,y)}{\partial y} = \frac{6y(2xy)-2x(4x^2+3y^2)}{(2xy)^2} = \frac{-8x^3+6xy^2}{4x^2y^2}$

Since f and f_y are continuous around (1,1),

by the existence and uniqueness theorem,the initial value problem has a unique solution.

(8 pts.) b)Find the unique solution of the initial value problem given above:

$\frac{dy}{dx} = \frac{4x^2+3^2}{2xy}$ is a homogenous equation.Let y=xv.Then,$x\frac{dv}{dx} = \frac{4+v^2}{2v}$.

Then,$\int \frac{2v}{4+v^2}dv = \int \frac{dx}{x}$.Then,$ln(4 + v^2) = ln(x) + ln(c)$.Hence,$4 + v^2 = cx$.

Therefore,$4 + \frac{y^2}{x^2} = cx$ is the general solution.Since $y(1) = 1, 4 + 1 = c \Rightarrow$

$c = 5$.Therefore,$4x^2 + y^2 = 5x^3$ is the unique solution.

(10 pts.)Q2. Find the orthogonal trajectories of the family of curves

$x^2 = ke^y + y + 1, k \in R$:

By the implicit differentiation,$2x = y'ke^y + y', k \in R$.Then,$\frac{dy}{dx} = y' = \frac{2x}{ke^y+1} = f(x, y)$.Since

$ke^y + 1 = x^2 - y, \frac{dy}{dx} = \frac{2x}{x^2-y}$.Then,$\frac{dy}{dx} = \frac{-1}{f(x,y)} = \frac{y-x^2}{2x}$ gives the orthogonal trajectories of family of curves.Then,$\frac{dy}{dx} - \frac{y}{2x} = \frac{-x}{2}$.Multiply both

sides by the integrating factor $\mu(x) = exp(\int \frac{-dx}{2x}) = \frac{1}{\sqrt{x}}$.

Then $\int d(\frac{y}{\sqrt{x}}) = \int -\frac{\sqrt{x}}{2}dx$.Then,$\frac{y}{\sqrt{x}} = -\frac{x^{3/2}}{3} + c$.

Therefore, $y = -\frac{x^2}{3} + c\sqrt{x}$ is the general solution.

(8 pts.) Q3. Consider the differential equation

$(2e^x + 3x^2y^{-1})dx + (2e^xy^{-1} - 3)dy = 0$.*Find an integrating factor in the form* $\mu = y^n$ *and then solve the equation.*

Let $M(x, y) = 2e^x + \frac{3x^2}{y}, N(x, y) = \frac{2e^x}{y} - 3$.

Suppose $y^nM(x, y)dx+y^nN(x, y)dy = 0$ is exact.Then,$\frac{\partial(y^nM(x,y))}{\partial y} = \frac{\partial(y^nN(x,y))}{\partial y}$

,or $2ny^{n-1}e^x + 3(n-1)x^2y^{n-2} = 2e^xy^{n-1}$

,or $(n-1)(2e^x)y^{n-1} + 3(n-1)x^2y^{n-2} = 0$,or n-1=0.

Therefore,$\mu(x,y) = y$ is an integrating factor for DE.

Thus,$(2\hat{e}^x y + 3x^2)dx + (2e^x - 3y)dy = 0$.

Then,$\frac{\partial F}{\partial x} = 2e^x y + 3x^2 \Rightarrow$

$F(x,y) = 2e^x y + x^3 + h(y)$.

Then,$\frac{\partial F}{\partial y} = 2e^x + h'(y) = N(x,y) = 2e^x - 3y \Rightarrow h'(y) = -3y \Rightarrow h(y) = \frac{-3y^2}{2} + c$.

Therefore,$2e^x y + x^3 - \frac{3y^2}{2} = c$ is the general solution.

(7 pts.) Q4. An object with a mass m is thrown upward from a ground level with initial velocity v_0.*Assume that the air resistance is*

proportional to the velocity.Using Newton's Law:Force equals to mass times accelaration,find:

(4 pts.) a) The velocity of the object at time $t > 0$.

We know that $a = \frac{dv}{dt}, F = ma = -mg - kv$,where k is the constant of

proportinality.Then,$m\frac{dv}{dt} = -mg - kv \Rightarrow \frac{dv}{dt} + \frac{k}{m}v = -g$.Then,by multiplying the integration factor $\mu(t,v) = e^{\frac{kv}{m}}$,we have

$\int d(e^{\frac{kt}{m}}v) = \int -ge^{\frac{-kt}{m}}dt \Rightarrow v = -\frac{mg}{k} + ce^{-\frac{kt}{m}}$.Since $v(0) = v_0$,we get $v_0 = -\frac{mg}{k} + c \Rightarrow c = (v_0 + \frac{mg}{k})$.

Therefore,$v = -\frac{mg}{k} + (v_0 + \frac{mg}{k})e^{-\frac{kt}{m}}$.

(3 pts.) b) The time t at which the object attains its maximum height.

If height is maximum at time t,then v(t)=0 $\Rightarrow 0 = -\frac{mg}{k} + (v_0 + \frac{mg}{k})e^{-\frac{kt}{m}} \Rightarrow t = ln((\frac{mg+v_0k}{mg})^{m/k})$.