EXERCISES IN NAIVE CALCULUS OF VARIATIONS

The following are simple exercises designed to help the student master the formal intricacies of the calculus of variations. $\star$ indicates p a t h o l o g y. The student should learn to cope with the great diversity of notation which is typical of the subject.

1. Find the critical functions for the following functionals:

(a) $\mathcal{F}[x] = \int_0^1 (x^2 + 2tx\dot{x})dt$ with $x(0) = 1$, $x(1) = 2$.

(b) $\mathcal{F}[x] = \int_0^1 (\dot{x}^2 + 12tx)dt$ with $x(0) = 1$, $x(1) = 3$.

(c) $\mathcal{F}[x] = \int_0^{\pi/2} (\dot{x}^2 + x^2)dt$ with $x(0) = 1$, $x(\pi/2) = 0$.

(d*) $\mathcal{F}[x] = \int_3^7 (\dot{x}^2)dt$ with $x(3) = 1$, $x(7) = 1$.

(e*) $\mathcal{F}[x] = \int_0^1 (\dot{x}^2 + t^2\dot{x})dt$ with $x(0) = 0$, $x(1) = 2$.

(f*) $\mathcal{F}[x] = \int_0^1 (\dot{x}^2 + t^2\dot{x})dt$ with $x(0) = 0$, $x(1) = 1$.

(g*) $\mathcal{F}[x] = \int_0^1 (x + t\dot{x})dt$ with $x(0) = 2$, $x(1) = 1$.

2. Write down the Euler-Lagrange equations for the following functionals:

(a) $\mathcal{F}[y] = \int_a^b (x^2 + 2y\frac{dy}{dx})dx$.

(b) $\mathcal{F}[y] = \int_a^b ((\frac{dy}{dx})^2 + 12x^3 \sin y)dx$.

(c) $\mathcal{F}[x, y] = \int_a^b (\dot{x}^2 + x^2 \sin y - t \cos \dot{y})dt$.

(d) $\mathcal{F}[x,y,z] = \int_a^b (x^3 + y\dot{z}^2 - \log(\dot{x}yz))dt$.

(e) $\mathcal{F}[\varphi,\psi] = \int_a^b \left[\varphi^2 + (\frac{d\varphi}{dx})^2\frac{d\psi}{dx} - \log(\varphi^2+\psi^2)\right] dx$.

(f) $\mathcal{F}[\varphi] = \int\int_\Omega \left[(\frac{\partial\varphi}{\partial x})^2) + (\frac{\partial\varphi}{\partial y})^2\right] dxdy$.

(g) $\mathcal{F}[\varphi] = \int\int_\Omega \left[y(\frac{\partial\varphi}{\partial x})^2) + x(\frac{\partial\varphi}{\partial y})^2\right] dxdy$.

(h) $\mathcal{F}[\varphi] = \int\int_\Omega \left[(\frac{\partial\varphi}{\partial x})^2 + (\frac{\partial\varphi}{\partial y})^2 + 2\rho\varphi\right] dxdy$ where $\rho = \rho(x,y)$ is a given function on Ω .

(i) $\mathcal{F}[\varphi] = \int\int\int_\Omega \left[(\frac{\partial\varphi}{\partial x})^2 + (\frac{\partial\varphi}{\partial y})^2 - (\frac{\partial\varphi}{\partial z})^2\right] dxdydz$.

Suggestions for Further Studies

Calculus of variations is a vast and beatiful subject of fundamental importance. A mathematician without at least a rudimentary knowledge thereof is a sad spectacle indeed.

There is a wide spectrum of excellent books written at different levels :

N. I. Akhiezer : *The Calculus of Variations*
G. A. Bliss : *Calculus of Variations*
L. E. Elsgolc : *Calculus of Variations*
O. Bolza : *Lectures on Calculus of Variations*
I. M. Gelfand, S. V. Fomin : *Calculus of Variations*

R. Weinstock : *Calculus of Variations with Applications to Physics and Engineering*
C. Lanczos : *The Variational Principles of Mechanics*
J. D. Logan : *Invariant Variational Problems*
D. Lovelock, H. Rund : *Tensors, Differential Forms and Variational Problems*

J. Marsden : *Applications of Global Analysis to Mathematical Physics*
M. Struwe : *Variational Methods*
J. Jost, X. Li-Jost : *Calculus of Variations*

R. Wodehouse : *A History of the Calculus of Variations in the 18th Century*
I. Todhunter : *A History of the Calculus of Variations in the 19th Century*