PROBLEMS (9)

1. For $a, b \in \mathbb{R}$, $0 < |b| < a$, prove that

$$\int_0^{2\pi} \frac{1}{a + b\cos\theta}\, d\theta = \frac{2\pi}{\sqrt{a^2 - b^2}}$$

$$\int_0^{2\pi} \frac{1}{(a + b\cos\theta)^2}\, d\theta = \frac{2\pi a}{(a^2 - b^2)^{3/2}}$$

$$\int_0^{2\pi} \frac{\cos\theta}{(a + b\cos\theta)^2}\, d\theta = -\frac{2\pi b}{(a^2 - b^2)^{3/2}}$$

$$\int_0^{2\pi} \frac{\cos^2\theta}{a + b\cos\theta}\, d\theta = \frac{2\pi a}{b^2}\left(\frac{a}{\sqrt{a^2 - b^2}} - 1\right)$$

2. For $a \in \mathbb{R} - \{\pm 1\}$ prove that

$$\int_0^{2\pi} \frac{d\theta}{1 - 2a\cos\theta + a^2} = \frac{2\pi}{|1 - a^2|} .$$

3. For $a, b \in \mathbb{R}$, $a, b > 0$ prove that

$$\int_0^{2\pi} \frac{d\theta}{a^2\cos^2\theta + b^2\sin^2\theta} = \frac{2\pi}{ab} .$$

4. For $a \in \mathbb{R}$ with $a > 0$ prove that

$$\int_0^{2\pi} \frac{d\theta}{a + \sin^2\theta} = \frac{2\pi}{\sqrt{a^2 + a}}$$

$$\int_0^{2\pi} \frac{d\theta}{(a + \sin^2\theta)^2} = \frac{2\pi(a + 1/2)}{(a^2 + a)^{3/2}} .$$

5. For any $n \in \mathbb{Z}$, $n \geq 0$ prove that

$$\int_0^{2\pi} \sin^{2n}\theta = \frac{\pi}{2^{2n-1}} \binom{2n}{n} .$$

6. Prove that

$$\int_0^\infty \frac{1}{x^4+x^2+1}\,dx = \frac{\pi}{2\sqrt{3}}$$
$$\int_0^\infty \frac{1}{(x^2+4x+5)^2}\,dx = \frac{\pi}{2}$$
$$\int_0^\infty \frac{1}{x^3+1}\,dx = \frac{2\pi}{3\sqrt{3}}$$
$$\int_0^\infty \frac{1}{x^4+1}\,dx = \frac{\pi}{\sqrt{2}}$$
$$\int_0^\infty \frac{x}{x^4+1}\,dx = \frac{\pi}{4}$$
$$\int_0^\infty \frac{1}{x^5+1}\,dx = \frac{4\pi}{5\sqrt{10-2\sqrt{5}}}$$
$$\int_0^\infty \frac{1}{x^6+1}\,dx = \frac{\pi}{3}$$

7. For $a, b \in \mathbb{R}$ with $a, b > 0$ prove that

$$\int_{-\infty}^\infty \frac{1}{(x^2+a^2)(x^2+b^2)}\,dx = \frac{1}{ab(a+b)}\,\pi$$
$$\int_{-\infty}^\infty \frac{1}{(x^2+a^2)^2(x^2+b^2)}\,dx = \frac{2a+b}{2a^3b(a+b)^2}\,\pi$$
$$\int_{-\infty}^\infty \frac{1}{(x^2+a^2)^2(x^2+b^2)^2}\,dx = \frac{a^2+3ab+b^2}{2a^3b^3(a+b)^3}\,\pi$$
$$\int_{-\infty}^\infty \frac{x^2}{(x^2+a^2)^2(x^2+b^2)^2}\,dx = \frac{1}{2ab(a+b)^3}\,\pi$$

8. Prove that

$$\int_{-\infty}^\infty \frac{x\sin x}{x^2+2x+2}\,dx = \frac{\pi}{e}(\cos 1 + \sin 1)$$

9. For $a, p \in \mathbb{R}$ with $a > 0$ prove that

$$\int_{-\infty}^\infty \frac{\cos(px)}{x^2+a^2}\,dx = \frac{\pi}{a}e^{-ap}$$
$$\int_{-\infty}^\infty \frac{\sin^2(px)}{x^2+a^2}\,dx = \frac{\pi}{2a}(1-e^{-2ap})$$

$$\int_{-\infty}^{\infty} \frac{x \sin(px)}{x^2+a^2}\, dx = \pi e^{-ap}$$

$$\int_{-\infty}^{\infty} \frac{\cos(px)}{x^4+a^4}\, dx = \frac{2\pi e^{-ap/\sqrt{2}}}{a^4} \left[\cos\left(\frac{ap}{\sqrt{2}}\right) + \sin\left(\frac{ap}{\sqrt{2}}\right)\right]$$

10. For $a, b \in \mathbb{R}$ with $a, b > 0$ prove that

$$\int_{-\infty}^{\infty} \frac{\cos(px)}{(x^2+a^2)(x^2+b^2)}\, dx = \begin{cases} -\dfrac{1}{b^2-a^2}\left(\dfrac{e^{-bp}}{b} - \dfrac{e^{-ap}}{a}\right)\pi & \text{if } a \neq b \\ \dfrac{ap+1}{2a^3}\, e^{-ap}\, \pi & \text{if } a = b \end{cases}$$

$$\int_{-\infty}^{\infty} \frac{x\sin(px)}{(x^2+a^2)(x^2+b^2)}\, dx = \begin{cases} \dfrac{e^{-bp} - e^{-ap}}{b^2-a^2}\,\pi & \text{if } a \neq b \\ -\dfrac{pe^{-ap}}{2a}\,\pi & \text{if } a = b \end{cases}$$

11. Prove that

$$\int_0^{\infty} \frac{\sin(\pi x)}{x(1-x^2)}\, dx = \pi$$

12. For any $p, q \in \mathbb{R}$ prove that

$$\int_{-\infty}^{\infty} \frac{\cos(px) - \cos(qx)}{x^2}\, dx = -\pi\ (p-q)$$

13. Prove that

$$\int_0^{\infty} \frac{\log x}{1+x^2}\, dx = 0$$

$$\int_0^{\infty} \frac{\log x}{(x^2+1)(x^2+4)}\, dx = \frac{-\pi \log 2}{12}$$

$$\int_0^{\infty} \frac{(\log x)^2}{1+x^2}\, dx = \frac{\pi^3}{8}$$

$$-\int_0^{\infty} \frac{\log x}{(1+x^2)^2}\, dx = \int_0^{\infty} \frac{x^2 \log x}{(1+x^2)^2}\, dx = \frac{\pi}{4}$$

14. Prove that for $a, b \in \mathbb{R}, b > 0$

$$\int_0^{\infty} \frac{\log x}{(x+a)^2+b^2}\, dx = \frac{\log(a^2+b^2)}{2b} \arctan\left(\frac{b}{a}\right).$$

15. Prove that

$$\begin{aligned}
\int_0^\infty \frac{1}{(1+x)\sqrt{x}}\,dx &= \pi \\
\int_0^\infty \frac{1}{(x^2+1)\sqrt{x}}\,dx &= \frac{\pi}{\sqrt{2}} \\
\int_0^\infty \frac{1}{(x^3+1)\sqrt{x}}\,dx &= \frac{2\pi}{3} \\
\int_0^\infty \frac{x^{1/3}}{(x^2+4x+8)^2}\,dx &= \sqrt{\frac{\pi}{3}}\,\sin\left(\frac{\pi}{12}\right) \\
\int_0^\infty \frac{\sqrt[3]{x}\log x}{x^2+1}\,dx &= \frac{\pi^2}{6}
\end{aligned}$$

16. Prove that for $\lambda \in \mathbb{R}$ with $0 < \lambda < 1$

$$\int_0^\infty \frac{x^{\lambda-1}\log x}{x+1}\,dx = \pi^2\cos(\pi\lambda)\ .$$

17. Prove that for $a, b \in \mathbb{R}$ with $-1 < a < 1$ and $0 < b < \pi$

$$\int_0^\infty \frac{x^a}{1+2x\cos b+x^2}\,dx = \frac{\pi}{\sin(a\pi)}\frac{\sin(ab)}{\sin b}\ .$$