

PROBLEMS (8)

1. (A) If $f : \Delta_1 = \{z \in \mathbb{C} \mid |z| < 1\} \rightarrow \Delta_1$ is an analytic function such that $f(a) = a$ for some $a \in \Delta_1$, prove that $|f'(a)| \leq 1$.

(B) If $f : \Delta_1 = \{z \in \mathbb{C} \mid |z| < 1\} \rightarrow \Delta_1$ is an analytic function with two distinct fixed points prove that f is the identity function.

(C) If $f : \Delta_1 = \{z \in \mathbb{C} \mid |z| < 1\} \rightarrow \Delta_1$ is an analytic function such that $f(a) = a$ for some $a \in \Delta_1$, and $f'(a) = 1$, prove that f is the identity function.

(D) Let $\Omega \subseteq \mathbb{C}$ be a simply connected open set. If $\Omega \neq \mathbb{C}$, prove that for each $a \in \Omega$ and $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, there exists a unique analytic bijection $g : \Omega \rightarrow \Omega$ such that $f(a) = a$ and $f'(a) = \lambda$.

2. Given $a, b, c, R_1, R_2 \in \mathbb{R}$ with $0 < R_1 \leq a \leq b \leq R_2$ and an analytic function $f : \Omega \rightarrow \mathbb{C}$ such that $\{z \in \mathbb{C} \mid R_1 \leq |z| \leq R_2\} \subseteq \Omega$ prove that

$$(\log c - \log a) \log M_f(b) \leq (\log c - \log b) \log M_f(a) + (\log b - \log a) \log M_f(c)$$

where $M_f(r) = \max \{|f(z)| \mid |z| = r\}$ for any $0 < R_1 \leq r \leq R_2$.

3. Let

$$\Omega = \mathbb{C} - \{xi \mid x \in \mathbb{R}, x^2 \geq 1\}.$$

(A) Let $h : \Omega \rightarrow \mathbb{C}$ be defined by

$$h(z) = \frac{2z}{1+z^2}$$

for any $z \in \Omega$. Prove that

$$f(\Omega_+) \subseteq \Omega_+$$

and

$$f(\Omega_-) \subseteq \Omega_-$$

where

$$\Omega_+ = \{z \in \mathbb{C} \mid \operatorname{Re}(z) > 0\} \quad \text{and} \quad \Omega_- = \{z \in \mathbb{C} \mid \operatorname{Re}(z) < 0\}.$$

(B) Prove that it is possible to define a branch of \arctan on Ω with $\arctan(0) = 0$.

(C) Prove that

$$h(z) = i \tan\left(2 \arctan(iz)\right)$$

for any $z \in \Omega$.

(D) Prove that

$$h^n(z) = i \tan\left(2^n \arctan(iz)\right)$$

for any $n \in \mathbb{Z}, n \geq 1$. for any $z \in \Omega$.

(E)¹ Prove that

$$\lim_{n \rightarrow \infty} h^n(z) = 1$$

¹E. Schröder : Über unendlich viele Algorithmen zur Auflösung der Gleichungen. *Mathematische Annalen* 2(1870)317-365 \rightarrow D. S. Alexander : *A History of Complex Dynamics*, p. 17.

if $z \in \Omega_+$ and

$$\lim_{n \rightarrow \infty} h^n(z) = -1$$

if $z \in \Omega_-$.

SUPPLEMENT TO PROBLEMS (6)

15. Given an analytic function $\Omega \subseteq \mathbb{C}$ and points

$$a_1, a_2, \dots, a_n \in \Delta_R = \{z \in \mathbb{C} \mid |z| < R\}$$

where $\overline{\Delta}_R \subseteq \Omega$, let

$$\tau(z) = \prod_{k=1}^n (z - a_k) .$$

Prove that

$$P(z) = \frac{1}{2\pi i} \int_{\Gamma_{0,R}} \frac{f(\zeta)}{\tau(\zeta)} \left(\frac{\tau(\zeta) - \tau(z)}{\zeta - z} \right) d\zeta$$

is a polynomial which coincides with f at a_1, a_2, \dots, a_n .

16. Let f be analytic on

$$\Delta_R = \{z \in \mathbb{C} \mid |z| < R\}$$

and continuous on $\overline{\Delta}_R$.

(A) Prove that

$$f(a) = \frac{1}{2\pi i} \int_{\Gamma(0,R)} \frac{f(z)}{z - a} dz$$

for any $a \in \Delta_R$.

(B) Prove that

$$f(a) = \frac{1}{2\pi i} \int_{\Gamma(0,R)} \frac{R^2 - |a|^2}{(z - a)(R^2 - z\bar{a})} f(z) dz$$

for any $a \in \Delta_R$.

(C) Prove the “ Poisson Formula ”

$$f(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr \cos(\theta - \varphi) + r^2} f(Re^{i\varphi}) d\varphi$$

for any $r \in [0, R)$ and any $\theta \in \mathbb{R}$.

(D) Suppose f satisfies $|f(z)| \geq 1$ for all $z \in \overline{\Delta}_R$. Applying the Poisson Formula to $\log f(z)$ prove that

$$|f(z)| \leq |f(0)|^{\frac{R+R'}{R-R'}}$$

when $|z| \leq R'$ with $0 < R' < R$.