P R O B L E M S (7)

1. Let $\Omega \subseteq \mathbb{C}$ be open and connected, $f : \Omega \longrightarrow \mathbb{C}$ be a nonconstant analytic function. If the absolute value of f attains a local minimum at $a \in \Omega$, prove that $f(a) = 0$.

2. (A) [1] Given a polynomial

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$

where $a_n \neq 0$, prove that

$$\frac{1}{2} \, | \, a_n z^n \, | \leq | \, p(z) \, | \leq \frac{3}{2} \, | \, a_n z^n \, |$$

for all $z \in \mathbb{C}$ with

$$|z| \geq \max \left(1 \ , \ \frac{2}{|a_n|} \sum_{k=0}^{n} | \, a_k \, | \right) .$$

(B) Prove that $p(\mathbb{C})$ is a closed set and deduce the fundamental theorem of algebra.

3. Find the maximum value of the absolute values of

$$(A) \ e^{z^2} \qquad (B) \ \frac{z+5}{z-5} \qquad (C) \ z^2 + z - 1 \qquad (D) \ 5 - |z|^2$$

on $K = \{ z \in \mathbb{C} \mid |z| \leq 1 \}$.

4.[2] If f is analytic on

$$\Delta_R = \{ z \in \mathbb{C} \mid |z| < R \}$$

with $f(0) = 1$ and $\mathrm{Re}(f(z)) > 0$ for all $z \in \Delta_R$, prove that

$$\frac{R - |z|}{R + |z|} \leq | \, f(z) \, | \leq \frac{R + |z|}{R - |z|}$$

for any $z \in \Delta_R$. (*Hint* : Consider $F = \varphi \circ f \circ \psi$ where ψ is an analytic map sending Δ_1 into Δ_R and φ is an analytic map sending the open right halfplane into Δ_1.)

5. If f is analytic on

$$\Delta_R = \{ z \in \mathbb{C} \mid |z| < R \}$$

and satisfies $f(\Delta_R) \subseteq \Lambda_h$ where

$$\Lambda_h = \{ z \in \mathbb{C} \mid \mathrm{Re}(z) < h \}$$

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[1] E. Freitag, R. Busam : *Funktionentheorie.*

[2] E. Copson : *An Introduction to the Theory of Functions of a Complex Variable.* Also some of the following problems in the similar vein.

for $h > 0$, prove that

$$|\mathrm{Im}(f(z))| \leq \frac{2h}{\pi} \frac{R+|z|}{R-|z|}$$

for any $z \in \Delta_R$. (*Hint* : Consider the bijection $\varphi : \{z \mid \mathrm{Re}(z) > 0\} \longrightarrow \Lambda_h$ defined by $\varphi(z) = \dfrac{2h}{\pi i} \log\left(\dfrac{1+z}{1-z}\right)$.)

6. If f is analytic in the upper half-plane

$$\Omega = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$$

and satisfies $f(\Omega) \subseteq \Omega$, prove that

$$\frac{|f(z) - f(a)|}{|f(z) - \overline{f(a)}|} \leq \frac{|z-a|}{|z-\bar{a}|}$$

for any $z, a \in \Omega$. Deduce that

$$|\ f'(z)\ | \leq \frac{\mathrm{Im}(f(z))}{\mathrm{Im}(z)}$$

for any $z \in \Omega$.

7. If f is analytic on

$$\Delta_R = \{z \in \mathbb{C} \mid |z| < R\}$$

and satisfies $|f(z)| \leq M$ for every $z \in \Delta_R$, prove that

$$\left| \frac{M\Big(f(z) - f(a)\Big)}{M^2 - \overline{f(a)}f(z)} \right| \leq \left| \frac{R(z-a)}{R^2 - \bar{a}z} \right|$$

for any $z, a \in \Delta_R$. Deduce that

$$\left(\frac{M}{R}\right) |f'(z)| \leq \frac{M^2 - |f(z)|^2}{R^2 - |z|^2}$$

for any $z \in \Delta_R$. (*Hint* : Find analytic $\varphi : \Delta_R \longrightarrow \Delta_1$ with $\varphi(a) = 0$.)

8. (A) Given analytic $\varphi : \Omega \subseteq_{op} \mathbb{C} \longrightarrow \mathbb{C}$, prove that for any $a \in \Omega$ the function $\psi : \Omega \longrightarrow \mathbb{C}$, defined by

$$\psi(z) = \begin{cases} \dfrac{\varphi(z) - \varphi(a) - (z-a)\varphi'(a)}{(z-a)^2} & \text{for} \quad z \neq a \\[2ex] \dfrac{1}{2}\varphi''(a) & \text{for} \quad z = a \end{cases}$$

is analytic on Ω .

(B) Consider an analytic function $f : \Delta_1 = \{z \in \mathbb{C} \mid |z| < 1\} \longrightarrow \mathbb{C}$ which satisfies $f(0) = f'(0) = 0$ and $|f(z)| \leq 1$ for all $z \in \Delta_1$. Prove that

$$\text{(B1)} \quad |f(z)| \leq |z|^2$$

for all $z \in \Delta_1 - \{0\}$ and

$$\text{(B2)} \qquad |f''(0)| \leq 2 \quad .$$

(C) Prove that in order for (B1) to reduce to an equality for even one $z \in \Delta_1 - \{0\}$ or (B2) to reduce to an equality it is necessary and sufficient that $f(z) = cz^2$ for some $c \in \mathbb{C}$ with $|c| = 1$.

(D) Let $g : \Delta_R = \{z \in \mathbb{C} \mid |z| < R\} \longrightarrow \mathbb{C}$ be an analytic function such that $|g(z)| \leq M$ for all $z \in \Delta_R$ and $g'(a) = 0$ for some $a \in \Delta_R$. Prove that

$$\left| \frac{M\Big(g(z) - g(a)\Big)}{M^2 - \overline{g(a)}g(z)} \right| \leq \left| \frac{R(z-a)}{R^2 - \bar{a}z} \right|^2$$

for all $z \in \Delta_R$ and

$$|g''(a)| \leq \frac{2M\Big(M^2 - |g(a)|^2\Big)}{R^2(R^2 - |a|^2)^2} \quad .$$

(*Hint :* Make use of the map $H_{r,b} : \Delta_r \longrightarrow \Delta_1$ for $r > 0$ and $b \in \Delta_r$ defined by

$$H_{r,b}(z) = \frac{r(z-b)}{r^2 - \bar{b}z}$$

and consider $f = H_{M,g(a)} \circ g \circ H_{R,a}^{-1}$.)

9.[3] Consider the analytic function $f : \Delta_1 = \{z \in \mathbb{C} \mid |z| < 1\} \longrightarrow \mathbb{C}$ which satisfies $f(0) = 0$ and $|f(z)| \leq 1$ for all $z \in \Delta_1$.

(A) Prove that

$$|f(z) + f(-z)| \leq 2|z|^2$$

for any $z \in \Delta_1$ and

$$|f''(0)| \leq 2 \ .$$

(B) Let $\omega = e^{2\pi i/3}$. Prove that

$$|f(z) + f(\omega z) + f(\omega^2 z)| \leq 3|z|^3$$

for any $z \in \Delta_1$ and

$$|f'''(0)| \leq 3 \ .$$

(C) What are the circumstances under which the any of the above inequalities turn into equalities ?

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[3] A. Pfluger : Varianten des Schwarzschen Lemma. *Elemente der Mathematik* 40 (**1985**46-47).

SUPPLEMENT TO PROBLEMS (6)

13. Let f be an entire function. If $\text{Re}(f)$ is bounded from above prove that f is a constant function. (*Hint* : Consider $e^{f(z)}$.) What happens if $\text{Re}(f)$ is not bounded from above but bounded from below ?

14. Let $f : \Omega \subseteq_{op,conn} \mathbb{C} \longrightarrow \mathbb{C}$ be an analytic function.

(A) If the equation $f(z) = c$ has infinitely many solutions in a compact subset of Ω for some $c \in \mathbb{C}$, prove that f is constant.

(B) Give an example of an analytic function with infinitely many roots in a closed subset of $\mathbb{C}$.

(C) Give an example of an analytic function with infinitely many roots in a bounded subset of $\mathbb{C}$.