

P R O B L E M S (6)

1. Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be the function defined by

$$f(z) = \begin{cases} 1 & \text{if } \operatorname{Im}(z) < 0 \\ 3 \operatorname{Im}(z) & \text{if } \operatorname{Im}(z) \geq 0 \end{cases}$$

and  $\gamma : [-1, +1] \rightarrow \mathbb{C}$  be the curve defined by  $\gamma(t) = t + t^3i$ . Prove that

$$\int_{\gamma} f(z) dz = \frac{7}{4} + \frac{5i}{2} .$$

2. If  $\gamma$  is the closed curve that traces out in counterclockwise fashion the square of vertices  $0, 1, 1 + i, i$  prove that

$$\int_{\gamma} \exp(\pi \bar{z}) dz = \frac{4}{\pi} (e^{\pi} - 1) .$$

3. Let  $\gamma$  be the curve that traces out in counterclockwise fashion the portion of the circle of center  $0 \in \mathbb{C}$  and radius 2 that lies in the first quadrant. Prove that

$$\left| \int_{\gamma} \frac{z^3}{z^4 \bar{z} - 1} dz \right| \leq \frac{4\pi}{31} .$$

4. Evaluate the following integrals :

(A)  $\int_{\gamma} (e^z - 4z) dz$  where  $\gamma : [0, \pi] \rightarrow \mathbb{C}$  with  $\gamma(t) = e^{it}$

(B)  $\int_{\gamma} \frac{dz}{z}$  where  $\gamma : [-1, 1] \rightarrow \mathbb{C}$  with  $\gamma(t) = -2t^2 + 1 + it$

(C)  $\int_{\gamma} e^z \cos z dz$  where  $\gamma : [0, 1] \rightarrow \mathbb{C}$  with  $\gamma(t) = 1 + t^5 + (1 - t^3 + t^6)\pi i$

(D)  $\int_{\gamma} z^{1/2} dz$  where  $\gamma : [-1, 1] \rightarrow \mathbb{C}$  with  $\gamma(t) = -t^2 + 1 + it$

5.(A) Prove that  $G : \mathbb{C} \rightarrow \mathbb{C}$  defined by

$$G(z) = \int_{-1}^2 \frac{\sin(tz)}{t} dt$$

for each  $z \in \mathbb{C}$  is an entire function and

$$G'(z) = \frac{\sin(2z) + \sin(z)}{z} .$$

(B) Find the power series expansion of  $G$  about  $z = 0$ .

6. Let  $\Omega \subseteq \mathbb{C}$  be open, connected. Clearly the set  $\mathfrak{A}(\Omega)$  of analytic functions on  $\Omega$  constitutes a commutative ring with identity under ordinary addition and multiplication. Prove that  $\mathfrak{A}(\Omega)$  is an integral domain.

7. The third derivative of an entire function  $f$  is bounded. Prove that  $f$  is a polynomial of degree at most 3.

8. Let  $f$  be an entire function. If  $\operatorname{Re}(f)$  is bounded from above prove that  $f$  is a constant function. (*Hint* : Consider  $e^{f(z)}$ .) What happens if  $\operatorname{Re}(f)$  is not bounded from above but bounded from below?

9. Let  $f$  be an entire function. If there exist  $\alpha, \beta \in \mathbb{C}$  with  $\alpha\bar{\beta} - \bar{\alpha}\beta \neq 0$ , such that

$$f(z + \alpha) = f(z + \beta) = f(z)$$

for every  $z \in \mathbb{C}$ , prove that  $f$  is constant.

10.(A) Given an entire function  $f$  which never attains the value 0, prove that there exists an entire function  $g$  such that  $f(z) = \exp(g(z))$  for all  $z \in \mathbb{C}$ . (*Hint* : Integrate  $f'/f$ .)

(B) Given entire functions  $c, s$  with the property that

$$(c(z))^2 + (s(z))^2 = 1$$

for all  $z \in \mathbb{C}$ , prove that there exists an entire function  $h$  such that

$$c = \cos \circ h, \quad s = \sin \circ h.$$

11. (A) Prove that the following assertions are equivalent for a non-constant entire function  $f$ :

(i)  $f(z) \rightarrow \infty$  as  $z \rightarrow \infty$ .

(ii)  $f(z)$  is bounded away from zero as  $z \rightarrow \infty$ .

(iii) There exists  $m \in \mathbb{Z}$  with  $m \leq 0$  such that  $z^m f(z) \rightarrow \infty$  as  $z \rightarrow \infty$ .

(iv) There exists  $m \in \mathbb{Z}$  with  $m \leq 0$  such that  $z^m f(z)$  is bounded away from zero as  $z \rightarrow \infty$ .

(v)  $f$  is a polynomial.

(B) Characterise entire functions  $f$  which satisfy  $f(f(z)) = z$  for all  $z \in \mathbb{C}$ .

12. Let  $f$  be an entire function which satisfies

$$f(z) + f(z + 1) = f(2z)$$

for all  $z \in \mathbb{C}$ .

(A) Show that

$$f(2^n z) = \sum_{k=0}^{2^n - 1} f\left(z + \frac{k}{2^{n-1}}\right)$$

for any  $n \in \mathbb{Z}$  with  $n \geq 1$   $z \in \mathbb{C}$  .

(B) Use the Cauchy integral formula over  $\Gamma(0, 2^n)$  to establish an estimate for  $|f''(z)|$  with  $z \in V(0, 1)$  in terms of

$$M = \sup_{z \in \bar{V}(0,3)} |f(z)|$$

(C) Prove that  $f(z) = az + b$  for some  $a, b \in \mathbb{C}$  with  $a + b = 0$  .