

P R O B L E M S (5)

1. (A) Given a series  $\sum a_n$  prove **d'Alembert's ratio test** :  $\sum a_n$  is absolutely convergent if

$$\limsup |a_{n+1}/a_n| < 1 .$$

- (B) Let  $a, b \in \mathbb{C}$  with  $0 < |a| < |b|$ . Prove that the series

$$1 + a + b^2 + a^3 + b^4 + \dots$$

is convergent iff it is absolutely convergent iff  $|b| < 1$  . This example demonstrates that d'Alembert's ratio test is weaker than the Cauchy-Hadamard Criterion.

2. Given a series  $\sum a_n$  with  $\lim_{n \rightarrow \infty} |a_{n+1}/a_n| = 1$ , prove **Raabe's Test** :  $\sum a_n$  is absolutely convergent if

$$\limsup \left[ n \left( \left| \frac{a_{n+1}}{a_n} \right| - 1 \right) \right] < -1 .$$

3. (A) Prove that a series  $\sum a_n$  is absolutely convergent if

$$\limsup \left( \frac{\log |a_n|}{\log n} \right) < -1 .$$

- (B) Prove that a series  $\sum a_n$  is not absolutely convergent if

$$\limsup \left( \frac{\log |a_n|}{\log n} \right) > -1 .$$

4. Find the set  $M$  of values of  $z \in \mathbb{C}$  for which the series

$$\sum \left( \frac{z}{1+z} \right)^n$$

is convergent. Is this series uniformly convergent on  $M$ ?

5. Find the disc of convergence of the following series :

(A)  $\sum n! z^n$  (B)  $\sum \frac{1}{n!} (z-5)^n$  (C)  $\sum z^{n!}$  (D)  $\sum z^{n^3}$  (E)  $\sum (5^n + i^n) (z-3)^n$

(F)  $\sum \frac{n!}{n^n} z^n$  (G)  $\sum \frac{(z-2)^n}{n\sqrt{n}}$  (H)  $\sum \frac{z^n}{n(\log n)^\mu}$  ( where  $\mu > 0$  )

(I)  $\sum \left( 1 - \frac{1}{n^2} \right)^{-n^5} z^n$  .

6. Let  $R > 0$  be the radius of convergence of  $\sum a_n z^n$ .  
 (A) Prove that the radius of convergence of  $\sum n^3 a_n z^n$  is equal to  $R$ .  
 (B) Express

$$\sum_{n=0}^{\infty} n^3 a_n z^n$$

in terms of

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

and its derivatives on  $V(0, R)$ .

7. For each  $n \in \mathbb{Z}, z \geq 0$ , let  $a_n$  be recursively defined by

$$a_{n+2} = \alpha a_{n+1} + \beta a_n$$

for some  $\alpha, \beta \in \mathbb{C}$  with  $a_0 = 0, a_1 = 1$ .

- (A) Prove that

$$|a_n| \leq \left[ 2 \max \left( |\alpha|, |\beta|, \frac{1}{2} \right) \right]^{n-1}.$$

- (B) Prove that the radius of convergence of  $\sum a_n z^n$  is

$$R = \min ( |z_1|, |z_2| )$$

where  $z_1, z_2$  are the roots of  $\beta z^2 + \alpha z - 1 = 0$ . Observe that  $R > 0$ .

- (C) Prove that for  $|z| < R$ ,

$$\sum_{n=0}^{\infty} a_n z^n = \frac{z}{1 - \alpha z - \beta z^2}.$$