PROBLEMS (4)

1. In each one of the following, express the function f in the form $u+iv$ by giving explicit formulae for u and v :

(A) $f(z) = \sin z$.
(B) $f(z) = \cos z$.
(C) $f(z) = \sinh z$.
(D) $f(z) = \cosh z$.
(E) $f(z) = \exp z^2$.
(F) $f(z) = z^3 + z$.

2. Prove that $f : \mathbb{C} \to \mathbb{C}$, defined by $f(z) = \bar{z}$ is nowhere differentiable .

3. Prove that $f : \mathbb{C} \to \mathbb{C}$, defined by $f(z) = x^3 + (1-y)^3 i$ is differentiable only at $z = i$. Evaluate $f'(i)$.

4. Prove that $f : \mathbb{C} \to \mathbb{C}$, defined by

$$f(z) = x^3y^2 + x^2y^3i$$

is differentiable exactly on the real and imaginary axes. Evaluate the derivative of f at such points.

5. Consider $f : \mathbb{C} \to \mathbb{C}$ defined by

$$f(z) = \begin{cases} \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2} & \text{for} \quad z \neq 0 \\ 0 & \text{for} \quad z = 0 \end{cases}$$

(A) Prove that f is a continous function,
(B) Prove that the real and imaginary parts of f satisfy the Cauchy-Riemann equations at $z = 0$.
(C) Prove that $f(z)$ is not differentiable at $z = 0$.

6. Prove that the real and imaginary parts of the function $f : \mathbb{C} \to \mathbb{C}$ defined by

$$f(z) = \sqrt{|xy|}$$

satisfy the Cauchy-Riemann equations at $z = 0$ but f is not differentiable at $z = 0$.

7. Prove that $f : \mathbb{C} \to \mathbb{C}$ defined by

$$f(z) = \begin{cases} \exp\left(\dfrac{1}{z^4}\right) & \text{for} \quad z \neq 0 \\ 0 & \text{for} \quad z = 0 \end{cases}$$

satisfies the Cauchy-Riemann equations at any $z \in \mathbb{C}$. Is $f(z)$ differentiable at $z = 0$?

8. Consider $f : \mathbb{C} \to \mathbb{C}$ defined by

$$f(z) = \begin{cases} \dfrac{x^3 y(y - xi)}{x^6 + y^2} & \text{for} \quad z \neq 0 \\ \\ 0 & \text{for} \quad z = 0 \end{cases}$$

Prove that

$$\frac{f(z) - f(0)}{z} \longrightarrow 0$$

as $z \to 0$ along any line through $z = 0$ but $f(z)$ is not differentiable at $z = 0$.

9.[1] Given the harmonic function $U : \Omega \subseteq_{op} \mathbb{C} \longrightarrow \mathbb{R}$ in each one of the following cases, construct a harmonic function $V : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{R}$ such that $f = U + iV : \Omega \longrightarrow \mathbb{C}$ is differentiable :

(A) $U = x^3 - 3xy^2 + 1 \ , \quad \Omega = \mathbb{C}$.

(B)

$$U = \frac{x}{x^2 + y^2} \ , \quad \Omega = \mathbb{C} - \{0\} \ .$$

(C) $U = e^x \Big(x \cos y - y \sin y \Big) \ , \quad \Omega = \mathbb{C}$.

(D)

$$U = \sqrt{\frac{x + \sqrt{x^2 + y^2}}{2}} \ , \quad \Omega = \mathbb{C} - (\infty, 0] \ .$$

(E)

$$U = \arctan\left(\frac{y}{x}\right) \ , \quad \Omega = \{ z \in \mathbb{C} \mid \operatorname{Re}(z) > 0 \} \ .$$

(F)

$$U = \frac{x \log (x^2 + y^2)}{2} - y \arctan\left(\frac{y}{x}\right) \ , \quad \Omega = \{ z \in \mathbb{C} \mid \operatorname{Re}(z) > 0 \} \ .$$

10.[2] Let $f = u + iv : \Omega \subseteq_{open} \longrightarrow \mathbb{C}$ be differentiable as a function of x, y at $a \in \Omega$ and

$$M_{\epsilon,a} = \left\{ \frac{f(z) - f(a)}{z - a} \quad \middle| \quad 0 < |z - a| < \epsilon \right\}$$

for any $\varepsilon > 0$ with $V(a, \varepsilon) \subseteq \Omega$. Prove that the set

$$\mathfrak{M}_a \quad = \bigcap_{V(a,\varepsilon) \subseteq \Omega} \overline{M}_{\epsilon,a}$$

is a circle of center

$$A = \frac{1}{2} \left[\Big(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \Big) + i \Big(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \Big) \right]_{z=a}$$

[1] E. Freitag, R. Busam : *Funktionentheorie.*

[2] A. I. Markouchevitch (Ed.) : *Fonctions d'une variable complexe. Problèmes contemporaines.*

and radius $|B|$ where

$$B = \frac{1}{2}\left[\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + i\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right]_{z=a} .$$

11.[3] Identifying the division ring $\mathbb{H}$ of quaternions with $\mathbb{R}^4$ as usual, consider a function

$$f : \Omega \subseteq_{op} \mathbb{H} \longrightarrow \mathbb{H}$$

where

$$f = U + V\mathsf{i} + W\mathsf{j} + Y\mathsf{k}$$

for some $U, V, W, Y : \Omega \subseteq_{op} \mathbb{H} \simeq \mathbb{R}^4 \longrightarrow \mathbb{R}$. f is said to be **differentiable on the right** at $\mathsf{a} \in \Omega$ if

$$\lim_{\mathsf{q} \longrightarrow \mathsf{a}} \Big(f(\mathsf{q}) - f(\mathsf{a})\Big)\Big(\mathsf{q} - \mathsf{a}\Big)^{-1}$$

exists. Similarly f is **differentiable on the left** at $\mathsf{a} \in \Omega$ if

$$\lim_{\mathsf{q} \longrightarrow \mathsf{a}} \Big(\mathsf{q} - \mathsf{a}\Big)^{-1}\Big(f(\mathsf{q}) - f(\mathsf{a})\Big)$$

exists.

(A) If f is differentiable on the right at $\mathsf{a} \in \Omega$, prove that

$$\begin{array}{ccccccc}
\dfrac{\partial U}{\partial x} & = & \dfrac{\partial V}{\partial y} & = & \dfrac{\partial W}{\partial z} & = & \dfrac{\partial Y}{\partial t} \\[2ex]
-\dfrac{\partial U}{\partial y} & = & \dfrac{\partial V}{\partial x} & = & -\dfrac{\partial W}{\partial t} & = & \dfrac{\partial Y}{\partial z} \\[2ex]
-\dfrac{\partial U}{\partial z} & = & \dfrac{\partial V}{\partial t} & = & \dfrac{\partial W}{\partial x} & = & -\dfrac{\partial Y}{\partial y} \\[2ex]
-\dfrac{\partial U}{\partial t} & = & -\dfrac{\partial V}{\partial z} & = & \dfrac{\partial W}{\partial y} & = & \dfrac{\partial Y}{\partial x}
\end{array}$$

at a .

(B) Show that $f : \mathbb{H} \longrightarrow \mathbb{H}$ defined by $f(\mathsf{q}) = \mathsf{q}^2$ is differentiable on the right only at $\mathsf{q} = 0$.

[3] C. A. Devaurs : The quaternion calculus. *American Mathematical Monthly* 80(**1973**)475-500