# PROBLEMS (3)

**1.** Prove that the points $z_1$, $z_2$, $z_3 \in \mathbb{C}$ are collinear iff

$$det \begin{bmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_2 \\ \bar{z}_1 & \bar{z}_2 & \bar{z}_3 \end{bmatrix} = 0 \ .$$

**2.** Prove that the area $\Delta$ of the triangle with vertices $a, b, c \in \mathbb{C}$ is

$$\Delta = det \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ \bar{a} & \bar{b} & \bar{c} \end{bmatrix} \ .$$

**3.** Prove that the lines $b\bar{z} + \bar{b}z + c = 0$ and $b'\bar{z} + \bar{b}'z + c' = 0$ and $b''\bar{z} + \bar{b}''z + c'' = 0$ are concurrent iff

$$det \begin{bmatrix} b & \bar{b} & c \\ b' & \bar{b}' & c' \\ b'' & \bar{b}'' & c'' \end{bmatrix} = 0 \ .$$

**4.** Prove that the points $z_1$, $z_2$, $z_3$, $z_4$ lie on the same circle or line iff

$$det \begin{bmatrix} 1 & 1 & 1 & 1 \\ z_1 & z_2 & z_3 & z_4 \\ \overline{z}_1 & \overline{z}_2 & \overline{z}_3 & \overline{z}_4 \\ z_1\overline{z}_1 & z_2\overline{z}_2 & z_3\overline{z}_3 & z_4\overline{z}_4 \end{bmatrix} = 0 \ .$$

**5.** (A) Prove that the perpendicular bisector of the line segment with endpoints $b$, $c \in \mathbb{C}$ is

$$(b - c)\bar{z} + (\bar{b} - \bar{c})z - (b\bar{b} - c\bar{c}) = 0 \ .$$

(B)Given distinct $b, c \in \mathbb{C}$ and $\lambda \in \mathbb{R}$, $\lambda \neq 1$, prove that the equation

$$|\frac{z - b}{z - c}| = \lambda$$

represents a circle whereof the center lies on the line containing $b, c$ . What happens if $\lambda = 1$?

**6.** Prove that triangles with vertices $a, b, c$ and $a', b', c'$ are (directly) similar iff

$$det \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a' & b' & c' \end{bmatrix} = 0 \ .$$

**7.** (A) Prove that $a, b, c \in \mathbb{C}$ constitute the vertices of an equilateral triangle iff

$$det \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ b & c & a \end{bmatrix} = 0 \ .$$

(B) Given $a, b \in \mathbb{C}$, prove that the following statements are equivalent :

(i) $0, a, b$ constitute the vertices of an equilateral triangle.
(ii) $a^2 + b^2 = ab$ .
(iii) $|a|^2 = |b|^2 = 2\mathrm{Re}(a\bar{b})$ .

(C) Prove that the roots of the equation

$$z^3 + Az^2 + Bz + C = 0$$

constitute the vertices of an equilateral triangle iff $A^2 = 3B$.

**8.** Prove that the equation of a parabola in $\mathbb{C}$ is

$$(a\bar{z} + \bar{a}z)^2 + 2(b\bar{z} + \bar{b}z) + c = 0$$

for some $a, b \in \mathbb{C}$ and $c \in \mathbb{R}$.

**9.**[1] Given $a_0, a_1, a_2, \in \mathbb{C}$ and $r_0, r_1, r_2 \in \mathbb{R}$ , let

$$\begin{aligned} \Delta_a &= a_0a_2 - a_1^2 \\ \Delta_r &= r_0r_2 - r_1^2 \\ H &= a_0r_2 - 2a_1r_1 + a_2r_0 \end{aligned}$$

and

$$\gamma(t) = \frac{a_0 + 2a_1t + a_2t^2}{r_0 + 2r_1t + r_2t^2}$$

for $t \in \mathbb{R}$.

(A) Prove that as t ranges over $\mathbb{R}$ $\gamma(t)$ traces out a hyperbola, parabola or an ellipse if $\Delta_r < 0, = 0$ or $> 0$, respectively.

(B) Show that in the last case the ellipse in question is a circle iff $4\Delta_a\Delta_r - H^2 = 0$.

**10.** (A) Making use of the van der Mond determinant

$$det \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} = (a-b)(b-c)(c-a)$$

or otherwise prove that a linear fractional transformation that leaves three <u>distinct</u> points of $\mathbb{C}P^1$ is the identity map.

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[1] R. Deaux : *Introduction to the Geometry of Complex Numbers*, pp. 95-97.

(B) Given distinct $a, b, c \in \mathbb{C}P^1$ prove that there exists a linear fractional transformation that maps $a, b, c$ into $0, 1, \infty \in \mathbb{C}P^1$ respectively.

(C) Given distinct $a, b, c \in \mathbb{C}P^1$ and distinct $a', b', c' \in \mathbb{C}P^1$ prove that there exists a unique linear fractional transformation that maps $a, b, c$ into $a', b', c'$ respectively.

(D) Construct a linear fractional transformation that maps the open unit disc $\Delta = \{z \in \mathbb{C} \mid |z| < 1\}$ onto the upper half plane $H = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ .

(E) Construct a linear fractional transformation that maps the open disc $\Delta = \{z \in \mathbb{C} \mid |z - 5| < 5\}$ onto the half plane $H = \{z \in \mathbb{C} \mid \mathrm{Re}(z) + \mathrm{Im}(z) > 7\}$ .

(F) Construct a linear fractional transformation that maps the crescent

$$\Gamma = \{z \in \mathbb{C} \mid |z+1|^2 > 2 \ , \ |z-1|^2 < 2\}$$

onto the " first quadrant " $H = \{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0 \ , \ \mathrm{Im}(z) > 0\}$ .

**11.** Let $\sigma : \mathbf{S}^2 \longrightarrow \mathbb{C}P^1$ be the stereographic projection.

(A) Prove that $\sigma^{-1}(z)$, $\sigma^{-1}(w) \in \mathbf{S}^2$ constitute the endpoints of a diameter iff $\{z, w\} \subseteq \mathbb{C}$ and $z\bar{w} = 1$ or $\{z, w\} = \{0, \infty\}$.

(B) Given $a, b \in \mathbb{C}$, suppose $\sigma^{-1}(a)$, $\sigma^{-1}(b) \in \mathbf{S}^2$ are the endpoints of a diameter. Prove that a rotation about this diameter through angle $\theta$ sends $\sigma^{-1}(z)$ into $\sigma^{-1}(w)$ iff

$$\frac{w-a}{w-b} = e^{i\theta} \ \frac{z-a}{z-b} \quad .$$

(C) Prove that a map $F : \mathbf{S}^2 \longrightarrow: \mathbf{S}^2$ is a rotation of $\mathbf{S}^2$ iff

$$\sigma \circ F \circ \sigma^{-1}(z) = \frac{az - \bar{b}}{bz - \bar{a}}$$

for some $a$, $b \in \mathbb{C}$ .