P R O B L E M S (1)

1. Prove that $\mathrm{Im}(iz) = \mathrm{Re}(z)$ and $\mathrm{Re}(iz) = -\mathrm{Im}(z)$ for any $z \in \mathbb{C}$.

2. Prove that $|z|^2 + a^2 = |z+a|^2 - 2\mathrm{Re}(az)$ for any $a \in \mathbb{R}$ and $z \in \mathbb{C}$.

3. Prove that $|z|^2 + 2\mathrm{Re}(bz) = |z+\bar{b}|^2 - |b|^2$ for any $b, z \in \mathbb{C}$.

4. Compute $(1+i)^n - (1-i)^n$ where $n \in \mathbb{Z}$, $n \geq 0$.

5. Putting $z = x + yi$ generically, compute

$$\mathrm{Re}(z^5) \quad , \quad \mathrm{Im}(z^{-2}) \quad , \quad \mathrm{Re}\Big(\frac{z-2}{z+2}\Big)$$

in terms of x, y .

6. Putting $z = x + yi$ generically, prove that

$$\frac{|x|+|y|}{\sqrt{2}} \ \leq \ |z| \ \leq \ |x| + |y| \ .$$

7. Putting $z = x + yi$ generically, prove that

$$\max(|x|,|y|) \ \leq \ |z| \ \leq \ \sqrt{2}\,\max(|x|,|y|) \ .$$

8. Given $c \in \mathbb{C}$ with $|c| = 1$ and $c \neq \pm 1$ prove that

$$\Big|\frac{z-c}{cz-1}\Big| = 1$$

iff $z \in \mathbb{R}$.

9. Prove that

$$|1-\bar{a}z|^2 - |z-a|^2 = (1-|z|^2)(1-|a|^2)$$

for any iff $z, a \in \mathbb{C}$.

10. Given $z, a \in \mathbb{C}$ with $|a| < 1$, prove that

$$\Big|\frac{z-a}{1-\bar{a}z}\Big| < 1, \quad = 1 \ , \quad > 1$$

iff $|z| < 1, \quad = 1 \ , \quad > 1$ respectively.

11. Prove that

$$|z|^2 + |w|^2 - |z^2 + w^2| \ \leq \ \frac{2|az+bw|^2}{a^2+b^2} \ \leq \ |z|^2 + |w|^2 + |z^2 + w^2|$$

for any $z, w \in \mathbb{C}$ and $a, b \in \mathbb{R}$ with $a^2 + b^2 \neq 0$.

(*Hint :* Introduce α with $\tan\alpha = a/b$ when $b \neq 0$ and express the middle term in the form $A + B\sin 2\alpha + C\cos 2\alpha$, evaluate minimum and maximum thereof.)

12.[1] (A) Given $c \in \mathbb{C} - \{0\}$, $d \in \mathbb{C}$, prove that

$$\frac{|c\bar{d} - \bar{c}d|}{2|c|} \leq |ct + d|$$

for any $t \in \mathbb{R}$.

(B) Let $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$, and $t \in \mathbb{R}$. If $c\bar{d} - d\bar{c} \neq 0$, prove that

$$\frac{|a\bar{b} - b\bar{a}|}{|ad - bc| + |a\bar{d} - b\bar{c}|} \leq |\frac{at + b}{ct + d}| \leq \frac{|ad - bc| + |a\bar{d} - b\bar{c}|}{|c\bar{d} - d\bar{c}|} .$$

What happens if $c\bar{d} - d\bar{c} = 0$?

13. For any $a, b, c \in \mathbb{R}$ with $a \neq 0, b^2 - 4ac < 0$, prove that the quotient ring

$$\mathfrak{R} = \frac{\mathbb{R}[x]}{\langle ax^2 + bx + c\rangle}$$

is a field which is isomorphic to $\mathbb{C}$. What happens if $b^2 - 4ac \geq 0$?

14.[2] For any field $\mathbb{F}$, let $\mathbb{F}^\times$ be the multiplicative group of nonzero elements of $\mathbb{F}$. Let $\mathbb{F}^+$ denote $\mathbb{F}$ as additive group.

(A) Prove that $\varphi : \mathbb{C}^\times \longrightarrow \mathbf{S}^1 = \{z \mid |z| = 1\}$ defined by $\varphi(z) = z/|z|$ is a group homorphism. Deduce that

$$\mathbf{S}^1 \simeq \mathbb{C}^\times/\mathbb{R}^\times_{>0}$$

where $\mathbb{R}^\times_{>0}$ is the multiplicative group of positive real numbers.

(B) Prove that $\psi : \mathbb{C}^\times \longrightarrow \mathbb{R}^+$ defined by $\psi(z) = \log|z|$ is a group homomorphism. Deduce that

$$\mathbb{R}^+ \simeq \mathbb{C}/\mathbf{S}^1$$

(C) Prove that

$$\mathbb{C}^\times \simeq \mathbb{R}^\times_{>0} \oplus \mathbf{S}^1$$

(D) Prove that

$$\mathbb{R} \simeq \mathbb{R}^\times_{>0}$$

(E) Express $\mathbb{R}$ as the direct sum of infinitely many copies of $\mathbb{Q}^+$ and prove that the groups $\mathbb{C}^\times$ and $\mathbf{S}^1$ are isomophic.

15. Consider $\mathbb{C}$ with its standard structure as a vector space over the field $\mathbb{R}$. The set $\{1, i\}$ constitutes the standard basis of $\mathbb{C}$ as vector space over $\mathbb{R}$.

(A) Prove that each $\mathbb{R}$-linear map $\varphi : \mathbb{C} \longrightarrow \mathbb{C}$ is of the form

$$\varphi(z) = az + b\overline{z}$$

[1] D. S. Mitrinović : E 1841. *American Mathematical Monthly* 72(**1965**)1129

[2] W. Horten, D. B. Shapiro : Proposal 1136. *Mathematics Magazine* 55(**1982**)44

for some $a, b \in \mathbb{C}$.

(B) Compute $\mathrm{tr}(\varphi)$, $\det(\varphi)$ and φ^{-1} if $\det(\varphi) \neq 0$ in terms of $a, b \in \mathbb{C}$.

(C) Prove that an $\mathbb{R}$-linear $\varphi : \mathbb{C} \longrightarrow \mathbb{C}$ is $\mathbb{C}$-linear iff $\varphi \circ J = J \circ \varphi$ where $J : \mathbb{C} \longrightarrow \mathbb{C}$ is an $\mathbb{R}$-linear map which is represented by the matrix

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

with respect to the standard basis of $\mathbb{C}$ over $\mathbb{R}$.

16. For any field $\mathbb{F}$, let $\mathbb{F}^{k\times k}$ denote the ring of k by k matrices with entries from $\mathbb{F}$.

(A) Prove that the subring of $\mathbb{R}^{2\times 2}$ consisting of matrices of the form

$$\begin{bmatrix} x & -y \\ y & x \end{bmatrix}$$

constitutes a field which is isomorphic to $\mathbb{C}$.

(B) Prove that $A \in \mathbb{R}^{2\times 2}$ satisfies $A^2 = -I$ iff there exists a non-singular $U \in \mathbb{R}^{2\times 2}$ such that

$$A = U \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} U^{-1}.$$

(*Hint* : Consider any $u \in \mathbb{R}^2 - \{(0,0)\}$ and note that $\{u, Au\}$ constitutes a basis for $\mathbb{R}^2$.)

(C) Prove that each $\mathbb{R}$-linear ring homomorphism $\psi : \mathbb{C} \longrightarrow \mathbb{R}^{2\times 2}$ is of the form

$$\psi(x+iy) = W \begin{bmatrix} x & -y \\ y & x \end{bmatrix} W^{-1}$$

for some non-singular $W \in \mathbb{R}^{2\times 2}$.

17. (A) Prove that

$$\mathbb{H} = \left\{ \begin{bmatrix} z & -w \\ \bar{w} & \bar{z} \end{bmatrix} \mid z, w \in \mathbb{C} \right\} \subseteq \mathbb{C}^{2\times 2}$$

constitutes a division ring with respect to the usual matrix addition and multiplication.

(B) Observe that each $\mathsf{q} \in \mathbb{H}$ can be written uniquely in the form

$$\mathsf{q} = x\mathsf{1} + y\mathsf{i} + z\mathsf{j} + w\mathsf{k}$$

for some $x, y, z, w \in \mathbb{R}$ where

$$\mathsf{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ \mathsf{i} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \ \mathsf{j} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \ \mathsf{k} = \begin{bmatrix} 0 & -i \\ -i & 1 \end{bmatrix}.$$

Prove that

$$\mathsf{i}^2 = \mathsf{j}^2 = \mathsf{k}^2 = -1$$

and

$$\mathsf{ij} = -\mathsf{ji} = \mathsf{k} \ , \ \mathsf{jk} = -\mathsf{kj} = \mathsf{i} \ , \ \mathsf{ki} = -\mathsf{ik} = \mathsf{j}$$

$\diamond$ The elements of the noncommutative division ring $\mathbb{H}$ are referred to as the **quaternions** [3]. Clearly $\mathbb{C}$ can be identified with $\{x + y\mathsf{i} \mid x, y \in \mathbb{R}\} \subset \mathbb{H}$ so that it is possible to obtain the division ring extensions

$$\mathbb{R} \subset \mathbb{C} \subset \mathbb{H}$$

and $\mathsf{1}$ may be omitted. The quaternions exhibit properties which seem at first to be similar to yet under closer scrutiny drastically deviate from those of complex numbers.$\diamond$

(C) Prove that $\mathbb{C} \times \mathbb{C}$ becomes a division ring under the multiplication

$$(z_1, w_1)(z_2, w_2) = (z_1 z_2 - w_1 \bar{w}_2, z_1 w_2 + w_1 \bar{z}_2)$$

which is isomorphic to $\mathbb{H}$.

(D) The **conjugate** $\bar{\mathsf{q}}$ of $\mathsf{q} \in \mathbb{H}$ is the complex conjugate of the matrix q. Prove that

$$\bar{\mathsf{q}} = x - y\mathsf{i} - z\mathsf{j} - w\mathsf{k}$$

if $\mathsf{q} = x + y\mathsf{i} + z\mathsf{j} + w\mathsf{k}$. Is conjugation a division ring automorphism in $\mathbb{H}$?

(E) The **norm** $|\mathsf{q}|$ of $\mathsf{q} \in \mathbb{H}$ is the positive square root of the determinant of of the matrix q. Prove that $|\mathsf{q}|^2 = \bar{\mathsf{q}}\mathsf{q} = \mathsf{q}\bar{\mathsf{q}}$ and $|\mathsf{q}\mathsf{r}| = |\mathsf{q}|\,|\mathsf{r}|$ for any $\mathsf{q}, \mathsf{r} \in \mathbb{H}$.

(F) Prove that

$$\begin{aligned}(a^2 + b^2 + c^2 + d^2)(x^2 + y^2 + z^2 + w^2) \;=\;& (ax - by - cz - dw)^2 \\ &+ (ay + bx + cw - dx)^2 \\ &+ (az + cx + dy - bw)^2 \\ &+ (aw + dx + bz - cy)^2\end{aligned}$$

for any $a, b, c, d, x, y, z, w \in \mathbb{R}$.

(G)[4] Given $\mathsf{q} \in \mathbb{H}$ define the **real part** $Re(\mathsf{q})$ of q by $2Re(\mathsf{q}) = \mathsf{q} + \bar{\mathsf{q}}$. Prove that

$$Re(\mathsf{q}) = \frac{1}{4}\Big(\mathsf{q} - \mathsf{iqi} - \mathsf{jqj} - \mathsf{kqk}\Big) .$$

[3] Quaternions were introduced by W. R. Hamilton in 1843.

[4] J. C. Holladay : A note on the Stone-Weierstrass theorem for quaternions. *Proceedings of the American Mathematical Society* 8(**1957**)656-657.