BASIC TENSOR ALGEBRA : PART 2

Every vector space U is isomorphic to its dual $U^\star$ simply because both spaces have the same dimension. However, this isomorphism is not canonical.

A bilinear map $\mathsf{g} \in \mathrm{Tens}^{0,2}(\mathrm{U})$ is a called an **inner product** on U if it is symmetric and positive definite. In the presence of an inner product $\mathsf{g} = \langle \bullet, \bullet \rangle$ on U there is a canonical isomorphism [1] between U and $U^\star$ that sends each $\mathsf{x} \in \mathrm{U}$ to $\mathsf{x}_\flat \in \mathrm{U}^\star$ defined by

$$\mathsf{x}_\flat = \langle \mathsf{x}, \bullet \rangle \quad .$$

The image of $\alpha \in \mathrm{U}^\star$ under the inverse of this isomorphism is denoted by $\alpha^\sharp \in \mathrm{U}$. This initiates a perfect symmetry between U and $U^\star$ so that

$$(\mathsf{x}_\flat)^\sharp = \mathsf{x} \quad \text{and} \quad (\alpha^\sharp)_\flat = \alpha$$

for any $\mathsf{x} \in \mathrm{U}$ and $\alpha \in \mathrm{U}^\star$. In this context the first move is to carry over the inner product $\mathsf{g} = \langle \bullet, \bullet \rangle$ on U to an inner product $\mathsf{g}^\star = \langle \bullet, \bullet \rangle^\star \in \mathrm{Tens}^{0,2}(\mathrm{U}^\star)$ on $U^\star$ defined by

$$\mathsf{g}^\star(\alpha, \beta) = \mathsf{g}(\alpha^\sharp, \beta^\sharp)$$

for any $\alpha, \beta \in \mathrm{U}^\star$. Clearly

$$\alpha^\sharp = \langle \alpha, \bullet \rangle^\star$$

and

$$\mathsf{g}(\mathsf{x}, \mathsf{y}) = \mathsf{g}^\star(\mathsf{x}_\flat, \mathsf{y}_\flat)$$

for any $\mathsf{x}, \mathsf{y} \in \mathrm{U}$.

In the presence of a basis $\{\mathsf{e}_i\}_{1 \le i \le n}$ for U and its dual $\{\mathsf{e}^i\}_{1 \le i \le n}$ for $U^\star$, if

$$\mathsf{g} = \langle \bullet, \bullet \rangle = g_{ij} \; \mathsf{e}^i \otimes \mathsf{e}^j$$

and

$$\mathsf{g}^\star = \langle \bullet, \bullet \rangle^\star = g^{ij} \; \mathsf{e}_i \otimes \mathsf{e}_j \; ,$$

clearly $(\mathsf{e}_i)_\flat = \langle \mathsf{e}_i, \bullet \rangle = g_{ij} \mathsf{e}^j$. Similarly $(\mathsf{e}^i)^\sharp = \langle \mathsf{e}^i, \bullet \rangle^\star = g^{ij} \mathsf{e}_j$.

In view of the fact that

$$\mathsf{e}_i = ((\mathsf{e}_i)_\flat)^\sharp = (g_{ip} \mathsf{e}^p)^\sharp = g_{ip} g^{pq} \mathsf{e}_q$$

[1] Folklorically referred to as the "musical correpondance " .

it is found that

$$g_{ip}g^{pj} = \delta_i^j$$

equivalently

$$[g^{pq}]_{1\leq p,q\leq n} = \left(\ [g_{ij}]_{1\leq i,j\leq n} \ \right)^{-1} \quad .$$

Concerning the change which the components of covectors and vectors under the "musical correspondence" there are the well establihed conventions of **index raising** and **index lowering** : If

$$\mathsf{x} = x^i \mathsf{e}_i,$$

then clearly

$$\mathsf{x}_\flat = x_i \mathsf{e}^i$$

where

$$x_i = g_{ip}\mathsf{e}^p.$$

Similarly, if

$$\alpha = a_i \mathsf{e}^i,$$

then clearly

$$\alpha^\sharp = a^i \mathsf{e}_i$$

where

$$a_i = g^{ip}\mathsf{e}_p.$$

The convention of index raising and lowering is directly extended to tensors other that vectors and covectors, with a little care exercised in tracing from where to where an index is raised or lowered. It has developed into a craft that is sometimes mockingly referred to as "index gymnastics" by the general mathematical public, but has proven itself quite indespensible in differential geometry, relativistic physics and continuum mechanics. A few examples will be sufficient to illustrate the basic idea :

$$g^{iw} \ g_{wj} = \delta^i_j \quad , \quad S_{ij}{}^k{}_{pq} = g^{kw} S_{ijwpq} = g_{qw} S_{ij}{}^k{}_p{}^w$$

PROBLEMS

I. Consider the n-dimensional Euclidean space or equivalently the inner product space $(\mathbb{R}^n, \langle , \rangle)$ where $\langle , \rangle = \delta_{ij} \ \mathsf{e}^i \otimes \mathsf{e}^j \in Tens^{(0,2)}(\mathbb{R}^n)$.

(A) Let $\mathsf{n} = n^i \mathsf{e}_i \in \mathbb{R}^n$ be a vector of unit length. Prove that the tensor

$$\mathsf{H} = (\delta_i{}^j - 2 \ n_i n^j) \ \mathsf{e}^i \otimes \mathsf{e}_j \in Tens^{(1,1)}(\mathbb{R}^n)$$

considered as a map $\mathsf{H} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is the reflection in the hyperplane perpendicular to n .

(B) Let $\mathsf{u} = u^i \mathsf{e}_i, \mathsf{v} = v^i \mathsf{e}_i \in \mathbb{R}^n$ be vectors of unit length with $\mathsf{u} \neq \mathsf{v}$. Prove that the tensor

$$\mathsf{K} = \left[\delta_i^{\ j} + \alpha \quad u_i v^j + v_i u^j - u_i u^j - v_i v^j \quad \right] \ \mathsf{e}^i \otimes \mathsf{e}_j \in Tens^{(1,1)}(\mathbb{R}^n)$$

where $\alpha = (1 - \langle \mathsf{u}, \mathsf{v} \rangle)^{-1}$, is a reflection that sends u into v.

II. Consider the 3-dimensional Euclidean space or equivalently the inner product space $(\mathbb{R}^3, \langle , \rangle)$ and let $\mathsf{a} = a^i \mathsf{e}_i \in \mathbb{R}^3$ be a vector of unit length.

(A) Prove that the tensor

$$\mathsf{R} = \left[\ \delta_i^{\ j} \cos\theta + a_i a^j (1 - \cos\theta) - \varepsilon_{i\ k}^{\ j} a^k \sin\theta \ \right] \ \mathsf{e}^i \otimes \mathsf{e}_j \in Tens^{(1,1)}(\mathbb{R}^3)$$

is the rotation about a, through θ .

(B) Relapse into matrix notation to prove that provided $\theta \neq \pi$

$$\mathsf{R} = \alpha \quad \beta\, v + \frac{1}{2} \mathsf{b}\mathsf{b}^T v + \mathsf{b} \times v$$

for any $v \in \mathbb{R}^3$ where

$$\mathsf{b} = 2 \tan \quad \frac{\theta}{2} \quad \mathsf{a}$$

$$\alpha = (1 + \frac{1}{4} \langle \mathsf{b}, \mathsf{b} \rangle)^{-1}$$

$$\beta = 1 - \frac{1}{4} \langle \mathsf{b}, \mathsf{b} \rangle \quad .$$

(C) Prove that

$$\varepsilon_{ijk} \varepsilon_{pqr} = \det \begin{bmatrix} \delta_{ip} & \delta_{iq} & \delta_{ir} \\ \delta_{jp} & \delta_{jq} & \delta_{jr} \\ \delta_{kp} & \delta_{kq} & \delta_{kr} \end{bmatrix}$$

(*Hint :* First handle the case $\varepsilon_{ijk} = \varepsilon_{ijk} \varepsilon_{123}$.)

(D) Let $\mathsf{u} = u^i \mathsf{e}_i, \mathsf{v} = v^i \mathsf{e}_i \in \mathbb{R}^3$ be vectors of unit length with $\mathsf{u} \neq -\mathsf{v}$. Prove that

$$\mathsf{R} = \left[\delta_i^{\ j} + v_i u^j - u_i v^j + \xi (u_i v^j - v_i u^j) + \eta (u_i u^j - v_i v^j)\right] \ \mathsf{e}^i \otimes \mathsf{e}_j$$

where

$$\xi = \frac{\langle \mathsf{u}, \mathsf{v} \rangle}{1 + \langle \mathsf{u}, \mathsf{v} \rangle}$$

$$\eta = \frac{1}{1 + \langle \mathsf{u}, \mathsf{v} \rangle}$$

is a rotation that sends u into v.

III. In classical terminology, a tensor $\mathsf{H} \in Tens(\mathbb{R}^n)$ is referred to as an **isotropic tensor** if $F[\mathsf{H}] = \mathsf{H}$ for all $F \in SO(n) = \{A \in \mathbb{R}^{n \times n} \mid A^T A = A A^T = I\}$.

(A) Prove that there exist no non-zero isotropic tensors of bidegree $(1, 0)$ or $(0, 1)$ over $\mathbb{R}^n$.

(B) Prove that $H \in Tens^{(0,2)}(\mathbb{R}^n)$ is isotropic iff

$$\mathsf{H}(Ax, y) + \mathsf{H}(x, Ay) = 0$$

for all $x, y \in \mathbb{R}^n$ and $A \in= \{A \in \mathbb{R}^{n \times n} \mid A^T + A = 0\}$. Derive similar conditions for tensors of bidegree $(1, 2)$ and $(0, 4)$ over $\mathbb{R}^n$.

(C) Prove that the only isotropic tensors of bidegree $(0, 2)$ over $\mathbb{R}^n$ are scalar multiples of $\langle , \rangle$.

(D) Prove that the only isotropic tensors of bidegree $(1, 2)$ over $\mathbb{R}^3$ are scalar multiples of the ordinary cross-product on $\mathbb{R}^3$, in other words, the tensor

$$\times = \varepsilon^i_{\ jk} \ \mathsf{e}_i \otimes \mathsf{e}^j \otimes \mathsf{e}^k \in Tens^{(1,2)}(\mathbb{R}^3) \quad .$$

(E) Prove that the subspace of isotropic tensors in $Tens^{(0,4)}(\mathbb{R}^n)$ is generated by tensors of the form

$$\delta_{ij}\ \delta_{kl}\ \mathsf{e}^i \otimes \mathsf{e}^j \otimes \mathsf{e}^k \otimes \mathsf{e}^l$$

and

$$(\delta_{ij}\ \delta_{kl} + \delta_{il}\ \delta_{jk})\ \mathsf{e}^i \otimes \mathsf{e}^j \otimes \mathsf{e}^k \otimes \mathsf{e}^l$$

and

$$(\delta_{ij}\ \delta_{kl} - \delta_{il}\ \delta_{jk})\ \mathsf{e}^i \otimes \mathsf{e}^j \otimes \mathsf{e}^k \otimes \mathsf{e}^l\ .$$