BASIC TENSOR ALGEBRA : PART 1

Consider an n-dimensional vector space U over $\mathbb{R}$, let $\mathrm{U}^\star$ be its dual. An element of U is called a **vector** and An element of $\mathrm{U}^\star$ is called a **covector**. $\mathrm{U}^{\star\star}$ is naturally identified with U, as usual. Let δ^i_j or δ^{ij} or δ_{ij} (it is at present immaterial whether the indices occur as subindices or superindices) be the **Kronecker's Delta** that stands for 0 if $i \neq j$ and for 1 if $i = j$. Given a basis $\{\mathsf{e}_i\}_{1\leq i\leq n}$ of U, the set $\{\mathsf{e}^i\}_{1\leq i\leq n}$ of covectors defined by $\mathsf{e}^i(\mathsf{e}_j) = \delta^i_j$ for $1 \leq i, j \leq n$ constitutes a basis for $\mathrm{U}^\star$. $\{\mathsf{e}_i\}_{1\leq i\leq n}$ and $\{\mathsf{e}^i\}_{1\leq i\leq n}$ are said to be **dual bases** for U and $\mathrm{U}^\star$ respectively. Given finite dimensional vector spaces U and V over $\mathbb{R}$ and a linear map $F : \mathrm{U} \longrightarrow \mathrm{V}$, the dual linear map $F^\star : \mathrm{V}^\star \longrightarrow \mathrm{U}^\star$ is defined by

$$F^\star(\beta)(\mathsf{x}) = \beta(F(\mathsf{x}))$$

for every $\beta \in \mathrm{V}^\star$ and $\mathsf{x} \in \mathrm{U}$.

Let $p, q \in \mathbb{Z}$, with $p, q \geq 0$. A **tensor** S of **bidegree** (p, q) over U is a multilinear map

$$\mathsf{S} \;:\; \underbrace{\mathrm{U}^\star \times \cdots \times \mathrm{U}^\star}_{p \text{ times}} \times \underbrace{\mathrm{U} \times \cdots \times \mathrm{U}}_{q \text{ times}} \longrightarrow \mathbb{R} \;.$$

A tensor of bidegree (p, q) over U is also described as being a tensor that is p-times **contravariant** and q-times **covariant** over U. The set of tensors of bidegree (p, q) over U has a natural structure as vector space which will be denoted by $\mathrm{Tens}^{p,q}(\mathrm{U})$. Similarly

$$\mathrm{Tens}(\mathrm{U}) = \bigcup_{p,q\in\mathbb{Z}} \mathrm{Tens}^{p,q}(\mathrm{U})$$

has a natural structure as vector space, albeit an infinite dimensional. Conventionally $\mathrm{Tens}^{0,0}(\mathrm{U})$ is identified with $\mathbb{R}$ and clearly $\mathrm{Tens}^{0,1}(\mathrm{U}) \simeq \mathrm{U}^\star$ whereas $\mathrm{Tens}^{1,0}(\mathrm{U}) \simeq \mathrm{U}^{\star\star} \simeq \mathrm{U}$. If $F : \mathrm{U} \longrightarrow \mathrm{V}$ is an isomorphism, then it has a natural extension

$$F[\bullet] : \mathrm{Tens}(\mathrm{U}) \longrightarrow \mathrm{Tens}(\mathrm{V})$$

defined for each $\mathsf{A} \in \mathrm{Tens}^{p,q}(\mathrm{U})$ by

$$F[\mathsf{A}](\beta^1, \ldots, \beta^p, \mathsf{y}_1, \ldots, \mathsf{y}_q) = \mathsf{A}(F^\star(\beta^1), \ldots, F^\star(\beta^p), F^{-1}(\mathsf{y}_1), \ldots, F^{-1}(\mathsf{y}_q))$$

for $\beta^i \in \mathrm{V}^\star$ and $y_j \in \mathrm{V}$ with $1 \leq i \leq p$ and $1 \leq j \leq q$.

An $\mathsf{F} \in \mathrm{Tens}^{1,r}(\mathrm{U})$ will be naturally interpreted as a multilinear map

$$\mathsf{F} : \underbrace{\mathrm{U} \times \cdots \times \mathrm{U}}_{r \text{ times}} \longrightarrow \mathrm{U}$$

sending $(\mathsf{x}_1, \ldots, \mathsf{x}_r)$ into $\mathsf{F}(\bullet, \mathsf{x}_1, \ldots, \mathsf{x}_r) \in \mathrm{U}^{\star\star} \simeq \mathrm{U}$. Conversely every multilinear map $\mathsf{F} : \underbrace{\mathrm{U} \times \cdots \times \mathrm{U}}_{r \text{ times}} \longrightarrow \mathrm{U}$ will be naturally interpreted as an element of $\mathrm{Tens}^{1,r}(\mathrm{U})$.

Given tensors $\mathsf{A} \in \mathrm{Tens}^{p,q}(\mathrm{U})$ and $\mathsf{B} \in \mathrm{Tens}^{p',q'}(\mathrm{U})$, the **tensor product** of A and B is the tensor $\mathsf{A} \otimes \mathsf{B} \in \mathrm{Tens}^{p+p',q+q'}(\mathrm{U})$ defined by

$$\mathsf{A} \otimes \mathsf{B}(\alpha^1, \ldots, \alpha^{p+p'}, \mathsf{x}_1, \ldots, \mathsf{x}_{q+q'}) = \\ \mathsf{A}(\alpha^1, \ldots, \alpha^p, \mathsf{x}_1, \ldots, \mathsf{x}_q)\mathsf{B}(\alpha^{p+1}, \ldots, \alpha^{p+p'}, \mathsf{x}_{q+1}, \ldots, \mathsf{x}_{q+q'})$$

for $\alpha^1, \ldots, \alpha^{p+p'} \in \mathrm{U}^\star$ and $\mathsf{x}_1, \ldots, \mathsf{x}_{q+q'} \in \mathrm{U}$. Tensor product is easily seen to be associative. However, it is not commutative. For instance, the covectors $\alpha, \beta \in \mathrm{U}^\star$ are linearly independent iff $\alpha \otimes \beta \neq \beta \otimes \alpha$. On the other hand, it can be easily checked that $\Omega \otimes \mathsf{A} = \mathsf{A} \otimes \Omega$ for every $\mathsf{A} \in \mathrm{Tens}^{p,0}(\mathrm{U})$ and $\Omega \in \mathrm{Tens}^{0,q}(\mathrm{U})$.

EXAMPLES : Obviously the concept of tensor product serves the purpose of building large tensors from out of smal tensors. Letting $\{\mathsf{e}_i\}_{1 \le i \le n}$ and $\{\mathsf{e}^i\}_{1 \le i \le n}$ stand for the standard dual bases in $\mathbb{R}^n \simeq \mathbb{R}^{n \times 1}$ and $(\mathbb{R}^n)^\star \simeq \mathbb{R}^{1 \times n}$ respectively, one can identify many familiar objects as tensors. For instance,

$$\langle \bullet, \bullet \rangle = \sum_{i,j} \delta_{ij} \ \mathsf{e}^i \otimes \mathsf{e}^j \in \mathrm{Tens}^{0,2}(\mathbb{R}^n)$$

is the ordinary inner product on $\mathbb{R}^n$ and

$$\mathrm{Id}_{\mathbb{R}^n} = \sum_{i,j} \delta^i_j \ \mathsf{e}_i \otimes \mathsf{e}^j \in \mathrm{Tens}^{1,1}(\mathbb{R}^n) \quad .$$

More generally, if $A : \mathrm{U} \longrightarrow \mathrm{U}$ is a linear transformation that sends e_j into $\sum_i A^i_j \ \mathsf{e}_i$, then

$$A = \sum_{i,j} A^i_j \ \mathsf{e}_i \otimes \mathsf{e}^j \in \mathrm{Tens}^{1,1}(\mathbb{R}^n) \quad .$$

Finally, defining **Eddington's epsilon** ε_{ijk} (or ε^i_{jk} it being at present immaterial whether the indices occur as subindices or superindices) by

$$\varepsilon_{ijk} = \begin{cases} 0 & \text{if} \quad \{i,j,k\} \neq \{1,2,3\} \\ 1 & \text{if} \quad \begin{bmatrix} 1 & 2 & 3 \\ i & j & k \end{bmatrix} \qquad \text{is even} \\ -1 & \text{if} \quad \begin{bmatrix} 1 & 2 & 3 \\ i & j & k \end{bmatrix} \qquad \text{is odd} \end{cases}$$

we find that

$$\times = \sum_{i,j,k} \varepsilon^i_{jk} \ \mathsf{e}_i \otimes \mathsf{e}^j \otimes \mathsf{e}^k \in \mathrm{Tens}^{1,2}(\mathbb{R}^3)$$

is the cross product on $\mathbb{R}^3$.

The above examples amply illustrate that many natural objects of geometry are tensors which can be broken down into constituents built out of vectors and covectors by means of tensor products. The following theorem shows that this is generally the case.

THEOREM : The set $\{\mathsf{e}_{i_1} \otimes \ldots \otimes \mathsf{e}_{i_p} \otimes \mathsf{e}^{j_1} \otimes \ldots \otimes \mathsf{e}^{j_q}\}_{1 \leq i_1, \cdots j_q \leq n}$ constitutes a basis for $\mathrm{Tens}^{p,q}(\mathrm{U})$.

Proof : The special case with $p = 1, q = 2$ will be sufficient to illustrate the idea : Take any $\mathsf{F} \in \mathrm{Tens}^{1,2}(\mathrm{U})$. Let $F^i_{jk} = \mathsf{F}(\mathsf{e}^i, \mathsf{e}_j, \mathsf{e}_k)$ for $1 \leq i, j, k \leq n$. Since

$$\left(\mathsf{F} - \sum_{i,j,k} F^i_{jk} \ \mathsf{e}_i \otimes \mathsf{e}^j \otimes \mathsf{e}^k\right)(\mathsf{e}^a, \mathsf{e}_b, \mathsf{e}_c) = F^a_{bc} - \sum_{i,j,k} F^i_{jk} \delta^a_i \delta^j_b \delta^k_c = 0$$

for all $1 \leq a, b, c \leq n$ we conclude that

$$\mathsf{F} = \sum_{i,j,k} F^i_{jk} \ \mathsf{e}_i \otimes \mathsf{e}^j \otimes \mathsf{e}^k \ .$$

Therefore the set $\{\mathsf{e}_i \otimes \mathsf{e}^j \otimes \mathsf{e}^k\}_{1 \leq i,j,k \leq n}$ spans the space $\mathrm{Tens}^{1,2}(\mathrm{U})$. On the other hand, if $\sum_{i,j,k} H^i_{jk} \ \mathsf{e}_i \otimes \mathsf{e}^j \otimes \mathsf{e}^k = 0$, then

$$H^a_{bc} = \left(\sum_{i,j,k} H^i_{jk} \ \mathsf{e}_i \otimes \mathsf{e}^j \otimes \mathsf{e}^k\right)(\mathsf{e}^a, \mathsf{e}_b, \mathsf{e}_c) = 0$$

for all a, b, c. Consequently the tensors $\mathsf{e}_i \otimes \mathsf{e}^j \otimes \mathsf{e}^k$ where $1 \leq i, j, k \leq n$ are linearly independent in $\mathrm{Tens}^{1,2}(\mathrm{U})$. ∎

It is now seen that in the presence of a basis $\{\mathsf{e}_i\}_{1 \leq i \leq n}$ for U and the dual basis $\{\mathsf{e}^i\}_{1 \leq i \leq n}$ for $U^\star$, each tensor $\mathsf{S} \in \mathrm{Tens}^{p,q}(\mathrm{U})$ can be uniquely expressed in the form

$$\mathsf{S} = \sum_{1 \leq i_1, \cdots j_q \leq n} S^{i_1 \ldots i_p}_{j_1 \ldots j_q} \ \mathsf{e}_{i_1} \otimes \ldots \otimes \mathsf{e}_{i_p} \otimes \mathsf{e}^{j_1} \otimes \ldots \otimes \mathsf{e}^{j_q} \ .$$

The numbers $S^{i_1 \ldots i_p}_{j_1 \ldots j_q} \in \mathbb{R}$ for $1 \leq i_1, \cdots j_q \leq n$ are called the **components** of the tensor S with respect to the basis $\{\mathsf{e}_i\}_{1 \leq i \leq n}$ for U. Clearly the components of a tensor are relative quantities that depend on the choice of the basis of the vector space U. If $\{\mathsf{f}_a\}_{1 \leq a \leq n}$ is another basis for U and $\{\mathsf{f}^a\}_{1 \leq i \leq n}$ is the dual basis for $U^\star$, then clearly

$$\mathsf{e}_i = \sum_{1 \leq a \leq n} M^a_i \mathsf{f}_a \quad , \quad \mathsf{e}^j = \sum_{1 \leq b \leq n} N^j_b \mathsf{f}^b$$

where $[M^a_i]_{1 \leq i,a \leq n}$ is a non-singular matrix and $[N^j_b]_{1 \leq j,b \leq n} = ([M^a_i]_{1 \leq i,a \leq n})^{-1}$. If $\tilde{S}^{i_1 \ldots i_p}_{j_1 \ldots j_q} \in \mathbb{R}$ for $1 \leq i_1, \cdots j_q \leq n$ are the components of the tensor S with respect to the basis $\{\mathsf{f}_a\}_{1 \leq a \leq n}$ for U, it can now be routinely checked that

$$\tilde{S}^{a_1 \ldots a_p}_{b_1 \ldots b_q} = \sum_{1 \leq i_1, \cdots j_q \leq n} M^{a_1}_{i_1} \ldots M^{a_p}_{i_p} \ N^{j_1}_{b_1} \ldots N^{j_q}_{b_q} \ S^{i_1 \ldots i_p}_{j_1 \ldots j_q}$$

CONVENTION : It must have been noticed that we have exercised great care in designating some of our indices as subscripts and the others as superscripts. This was done in preparation for the introduction of a convention ascribed to A. Einstein. Although it is by no means mandatory, the use of this convention results in enormous brevity and simplicity in the computations with tensors. Basically it consists in carefully omitting the sigmas denoting summation and stipulating that summation is to be effected with respect to indices that occur twice in a term, once as subscript, once as superscript, unless an explicit forewarning to the contrary is issued. An index occuring only once in a term is called a **free index.** In this convention the occurence of an index twice as subcript or as superscript in the same term as well as any kind of occurences of the same index more than twice in the same term are excluded. To illustrate the idea, some of the tensors or components of tensors that occured above can be rewritten as follows :

$$\begin{aligned} \langle \bullet, \bullet \rangle &= \delta_{ij}\ \mathsf{e}^i \otimes \mathsf{e}^j \\ \mathrm{Id}_{\mathbb{R}^n} &= \delta_i^j\ \mathsf{e}^i \otimes \mathsf{e}_j = \mathsf{e}^i \otimes \mathsf{e}_i \\ \times &= \varepsilon^i_{jk}\ \mathsf{e}_i \otimes \mathsf{e}^j \otimes \mathsf{e}^k \\ \mathsf{F} &= F^i_{jk}\ \mathsf{e}_i \otimes \mathsf{e}^j \otimes \mathsf{e}^k. \\ \mathsf{S} &= S^{i_1 \dots i_p}_{j_1 \dots j_q}\ \mathsf{e}_{i_1} \otimes \dots \otimes \mathsf{e}_{i_p} \otimes \mathsf{e}^{j_1} \otimes \dots \otimes \mathsf{e}^{j_q}\ . \end{aligned}$$

$$\tilde{S}^{a_1 \dots a_p}_{b_1 \dots b_q} = M^{a_1}_{i_1} \dots M^{a_p}_{i_p}\ N^{j_1}_{b_1} \dots N^{j_q}_{b_q}\ S^{i_1 \dots i_p}_{j_1 \dots j_q}$$

In the sequel we shall consistently employ the **Einstein Summation Convention** .

Given $p, q \geq 1$, the **contraction** of $\mathsf{A} \in \mathrm{Tens}^{p,q}(\mathrm{U})$ along the rth and the sth arguments where $1 \leq r \leq p$ and $1 \leq s \leq q$, is the tensor $C^r_s\mathsf{A} \in \mathrm{Tens}^{p-1,q-1}(\mathrm{U})$ defined by

$$C^r_s\mathsf{A}(\alpha^1, \dots, \alpha^{r-1}, \alpha^{r+1}, \dots, \alpha^p, \mathsf{x}_1, \dots, \mathsf{x}_{s-1}, \mathsf{x}_{s+1}, \dots, \mathsf{x}_q) =$$
$$\mathsf{A}(\alpha^1, \dots, \alpha^{r-1}, \mathsf{e}^a, \alpha^{r+1}, \dots, \alpha^p, \mathsf{x}_1, \dots, \mathsf{x}_{s-1}, \mathsf{e}_a, \mathsf{x}_{s+1}, \dots, \mathsf{x}_q)$$

where $\{\mathsf{e}_i\}_{1 \leq i \leq n}$ and $\{\mathsf{e}^i\}_{1 \leq i \leq n}$ are dual bases for U and $\mathrm{U}^\star$ respectively. Note that this definition is independent of the choice of the dual bases. It is easy and illuminating to write contractions in terms of components : Given $\mathsf{A} \in \mathrm{Tens}^{p,q}(\mathrm{U})$ with components $A^{i_1 \dots i_p}_{j_1 \dots j_q}$, its contraction $C^r_s\mathsf{A} \in \mathrm{Tens}^{p-1,q-1}(\mathrm{U})$ along the rth and the sth arguments where $1 \leq r \leq p$, $1 \leq s \leq q$, has components

$$A^{i_1 \dots i_{r-1} a i_{r+1} \dots i_p}_{j_1 \dots j_{s-1} a j_{s+1} j_q}\ .$$

It should be noticed that the operation of contraction is really a direct generalisation of the operation of taking traces of linear operators : Indeed, if $A : \mathrm{U} \longrightarrow \mathrm{U}$ is a linear map which sends e_i into $A^j_i\mathsf{e}_j$ so that $A = A^j_i\mathsf{e}_j \otimes \mathsf{e}^i$,

THEOREM : Given an isomorphism $F : \mathrm{U} \longrightarrow \mathrm{V}$, the map $F[\bullet] : \mathrm{Tens}(\mathrm{U}) \longrightarrow \mathrm{Tens}(\mathrm{V})$ respects the products and contractions of of tensors. To be precise,

$$F[\mathsf{A} \otimes \mathsf{B}] = F[\mathsf{A}] \otimes F[\mathsf{B}]$$

for any $\mathsf{A}, \mathsf{B} \in \mathrm{Tens}(\mathrm{U})$ and

$$F[C^r_s\mathsf{H}] = C^r_s F[\mathsf{H}]$$

for any $\mathsf{H} \in \mathrm{Tens}^{p,q}(\mathrm{U})$ where $p, q \in \mathbb{Z}$, with $p, q \geq 0$ and $1 \leq r \leq p$, $1 \leq s \leq q$.

Proof : Again, it is quite sufficient to illustrate the idea on low dimensional instances. Suppose that $\mathsf{A} \in \mathrm{Tens}^{2,1}(\mathrm{U})$ and $\mathsf{B} \in \mathrm{Tens}^{0,2}(\mathrm{U})$. Clearly

$$\begin{aligned} F[\mathsf{A}\otimes\mathsf{B}](\beta^1,\beta^2,\mathsf{y}_1,\mathsf{y}_2,\mathsf{y}_3) &= \mathsf{A}\otimes\mathsf{B}(F^\star(\beta^1),F^\star(\beta^2),F^{-1}(\mathsf{y}_1),F^{-1}(\mathsf{y}_2),F^{-1}(\mathsf{y}_3)) \\ &= \mathsf{A}(F^\star(\beta^1),F^\star(\beta^2),F^{-1}(\mathsf{y}_1))\mathsf{B}(F^{-1}(\mathsf{y}_2),F^{-1}(\mathsf{y}_3)) \\ &= F[\mathsf{A}](\beta^1,\beta^2,\mathsf{y}_1)F[\mathsf{B}](\mathsf{y}_2,\mathsf{y}_3) \\ &= F[\mathsf{A}]\otimes F[\mathsf{B}](\beta^1,\beta^2,\mathsf{y}_1,\mathsf{y}_2,\mathsf{y}_3) \end{aligned}$$

for any $\beta^1, \beta^2 \in \mathrm{U}^\star$ and $y_1, y_2, y_3 \in \mathrm{U}$. Now consider $\mathsf{H} \in \mathrm{Tens}^{3,4}(\mathrm{U})$. Clearly

$$\begin{aligned} F[C^2_3\mathsf{H}](\beta^1,\beta^3,\mathsf{y}_1,\mathsf{y}_2,\mathsf{y}_4) &= C^2_3\mathsf{H}(F^\star(\beta^1),F^\star(\beta^3),F^{-1}(\mathsf{y}_1),F^{-1}(\mathsf{y}_2),F^{-1}(\mathsf{y}_4)) \\ &= \mathsf{H}(F^\star(\beta^1),\mathsf{e}^a,F^\star(\beta^3),F^{-1}(\mathsf{y}_1),F^{-1}(\mathsf{y}_2),\mathsf{e}_a,F^{-1}(\mathsf{y}_4)) \\ &= \mathsf{H}(F^\star(\beta^1),F^\star(\mathsf{f}^a),F^\star(\beta^3),F^{-1}(\mathsf{y}_1),F^{-1}(\mathsf{y}_2),F^{-1}(\mathsf{f}_a),F^{-1}(\mathsf{y}_4)) \end{aligned}$$

where $\mathsf{f}_a = F(\mathsf{e}_a)$ and $\mathsf{f}^a = (F^\star)^{-1}(\mathsf{e}^a)$. Since $\{\mathsf{f}_a\}_{1\leq a\leq n}$ and $\{\mathsf{f}^a\}_{1\leq a\leq n}$ constitute dual bases for U and $\mathrm{U}^\star$ respectively, we conclude that

$$F[C^2_3\mathsf{H}](\beta^1,\beta^3,\mathsf{y}_1,\mathsf{y}_2,\mathsf{y}_4) = C^2_3F[\mathsf{H}](\beta^1,\beta^3,\mathsf{y}_1,\mathsf{y}_2,\mathsf{y}_4)$$

for all $\beta^1, \beta^3 \in \mathrm{U}^\star$ and $y_1, y_2, y_4 \in \mathrm{U}$. ∎

PROBLEMS

I. Express the following elements of $Tens^{(1,p)}(\mathbb{R}^n)$ as multilinear maps from $(\mathbb{R}^n)^p$ to $\mathbb{R}^n$ by describing their effect on the standard bases :

(A) $2\mathsf{e}_2\otimes\mathsf{e}^1 + 3\mathsf{e}_1\otimes\mathsf{e}^2 - 2\mathsf{e}_1\otimes\mathsf{e}^1 \in Tens^{(1,1)}(\mathbb{R}^2)$.

(B) $3\mathsf{e}_1\otimes\mathsf{e}^2\otimes\mathsf{e}^2 - 5\mathsf{e}_2\otimes\mathsf{e}^2\otimes\mathsf{e}^1 \in Tens^{(1,2)}(\mathbb{R}^2)$.

II. Given $\mathsf{x},\mathsf{y} \in \mathrm{U} \simeq Tens^{1,0}(\mathrm{U})$, $\lambda \in \mathrm{U}^\star \simeq Tens^{0,1}(\mathrm{U})$, $\mathsf{H} \in Tens^{0,2}(\mathrm{U})$ and $\mathsf{K} \in Tens^{1,1}(\mathrm{U})$ prove that

(A) $C^1_1(\lambda\otimes\mathsf{x}) = \lambda(\mathsf{x}) \in \mathbb{R}$

(B) $C^1_1(\mathsf{H}\otimes\mathsf{y}) = \mathsf{H}(\mathsf{y},\bullet) \in Tens^{0,1}(\mathrm{U}) \simeq \mathrm{U}^\star$.

(C) $C^1_1((C^1_1(\mathsf{H}\otimes\mathsf{x}))\otimes\mathsf{y}) = \mathsf{H}(\mathsf{x},\mathsf{y}) \in \mathbb{R}$

(D) $C^1_1((C^1_2(\mathsf{H}\otimes\mathsf{x}))\otimes\mathsf{y}) = \mathsf{H}(\mathsf{y},\mathsf{x}) \in \mathbb{R}$.

(E) $C^1_1((C^1_2(\mathsf{K}\otimes\lambda))\otimes\mathsf{x}) = \mathsf{K}(\lambda,\mathsf{x}) \in \mathbb{R}$.