PROBLEMS

I. The vector field

$$A = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} \in \mathfrak{X}(\mathbb{R}^2)$$

satisfies

$$\mathrm{L}_A(y\ dx + x\ dy) = 0$$

iff

$$Q + \frac{\partial P}{\partial x}y + \frac{\partial Q}{\partial x}x = 0$$

$$P + \frac{\partial P}{\partial y}y + \frac{\partial Q}{\partial y}x = 0\ .$$

II. The vector field

$$A = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} \in \mathfrak{X}(\mathbb{R}^2)$$

is divergence free iff it satisfies

$$\mathrm{L}_A(dx \otimes dy - dy \otimes dx) = 0\ .$$

III. Consider the Riemannian metric tensor field

$$\mathsf{g} = \alpha\ dx \otimes dx + F(x)\ \ (\ \beta\ dy \otimes dy + \gamma\ dz \otimes dz\ )$$

on $\mathbb{R}^3$ where α, β, γ are constants. Prove that the vector space of Killing vector fields of g is generated by the vector fields

$$\frac{\partial}{\partial y}\ ,\ \frac{\partial}{\partial z}\quad \text{and}\quad \gamma\ z\frac{\partial}{\partial y} - \beta\ y\frac{\partial}{\partial z}\quad .$$

IV. Consider the Riemannian metric tensor field

$$\mathsf{g} = \alpha\ dx \otimes dx + F(x)\ \ (\ \beta\ dy \otimes dy + \gamma\ dz \otimes dz + \delta\ dt \otimes dt\ )$$

on $\mathbb{R}^4$ where α, β, γ, δ are constants. Prove that the vector space of Killing vector fields of g is generated by the vector fields

$$\frac{\partial}{\partial y}\ ,\ \frac{\partial}{\partial z}\ ,\ \frac{\partial}{\partial t}$$

and

$$\gamma\ z\frac{\partial}{\partial y} - \beta\ y\frac{\partial}{\partial z},\ \delta\ t\frac{\partial}{\partial z} - \gamma\ z\frac{\partial}{\partial t},\ \beta\ y\frac{\partial}{\partial t} - \delta\ t\frac{\partial}{\partial y}\quad .$$

V. Consider the Riemannian metric tensor field

$$\mathsf{g} = F(t)\ \ dx \otimes dx + e^{2x}\,(dy \otimes dy + dz \otimes dz)\ \ + \delta\ dt \otimes dt$$

on $\mathbb{R}^4$ where α, β, γ, δ are constants. Prove that the vector space of Killing vector fields of g is generated by the vector fields

$$\frac{\partial}{\partial y}\ ,\ \frac{\partial}{\partial z}\ ,\ z\frac{\partial}{\partial y} -\ y\frac{\partial}{\partial z}\ ,\ \ -\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} +\ z\frac{\partial}{\partial z}$$

$$y\,\frac{\partial}{\partial x}+\frac{1}{2}\;y^2-z^2-e^{-2x}\;\frac{\partial}{\partial y}+yz\frac{\partial}{\partial z}\quad,\quad z\,\frac{\partial}{\partial x}+\frac{1}{2}\;z^2-y^2-e^{-2x}\;\frac{\partial}{\partial z}+zy\frac{\partial}{\partial y}\quad .$$

VI. (A) Prove that for any tensor field Q on M and $A, B \in \mathfrak{X}(M)$,

$$\mathrm{L}_{[A,B]}\mathsf{Q} = \mathrm{L}_A\mathrm{L}_B\mathsf{Q} - \mathrm{L}_B\mathrm{L}_A\mathsf{Q}$$

(*Hint* : Establish for small tensors and proceeed by induction via the Leibniz Rule)

(B) Let (M, g) be an n-dimensional Riemannian manifold. If $X \in \mathfrak{X}(M)$ is a conformal vector field, prove that

$$\mathrm{L}_X\;\mathsf{g} = \frac{2}{n}\;\mathrm{div}(X)\;\mathsf{g}\quad .$$

(C) If $A, B \in \mathfrak{X}(M)$ are conformal vector fields, prove that $[A, B]$ is a conformal vector field and

$$\mathrm{div}([A, B]) = A\mathrm{div}(B) - B\mathrm{div}(A)$$

VII. Consider $f \in \mathfrak{F}(M)$, $A \in \mathfrak{X}(M)$.

(A) Prove that

$$\mathrm{L}_A df = d(Af)\ .$$

(B) Prove that

$$\mathrm{L}_A\mathrm{grad}(f) = \mathrm{grad}(Af)\ .$$

VIII. Let (M, g) be an orientable Riemannian manifold with volume form Vol_{g} . Prove that

$$\mathrm{L}_A Vol_{\mathsf{g}} = \mathrm{div}(A)Vol_{\mathsf{g}} = d(A \pitchfork Vol_{\mathsf{g}}).$$

IX. Consider the Riemannian manifold (M, g), let $U \in \mathfrak{X}(M)$ be a unit vector field.

(A) Prove that $\mathsf{N}_U : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$ defined by

$$\begin{aligned}\mathsf{N}_U(X, Y) &= \mathrm{L}_X\mathsf{g}(U, Y) - \mathrm{L}_Y\mathsf{g}(U, X) + \mathsf{g}(X, [U, Y]) + \mathsf{g}(Y, [U, X]) \\ &\quad + \mathrm{L}_U\mathsf{g}(U, X)\;\mathsf{g}(U, Y) - \mathrm{L}_U\mathsf{g}(U, Y)\;\mathsf{g}(U, X) + \mathsf{g}(U, [X, Y])\quad .\end{aligned}$$

is a tensor field.

(B) Prove that $\mathsf{N}_U \equiv 0$ iff

$$U = \frac{\mathrm{grad}(\varphi)}{\|\;\mathrm{grad}(\varphi)\;\|}$$

where $\varphi \in \mathfrak{F}(M)$ is a scalar field with non-vanishing gradient. .