Genaral Topology , 2008 Fall

PROBLEMS (3)

1. Let (X, d) be a metric space. Prove that each open set in X can be expressed as a union of sets from the family $\mathfrak{C}$ where

$$\mathfrak{C} = \{B_d[a, R] \mid a \in X, R > 0\} \ .$$

2. Let (X, d) , (X, d) , be metric spaces. Prove that the topology of $X \times Y$ is exactly the topology induced by the metric Δ on $X \times Y$ defined by

$$\Delta((x, y), (x', y')) = \max\Big(d(x, x'), e(y, y')\Big)$$

for every $x, x' \in X$ and $y, y' \in Y$.

3. Prove that a sepable metric space is second countable.

4. Let $(X, \geq)$ be a linearly ordered set. The *right order topology* on X is the topology which admits as a basis the family $\mathfrak{R} = \{R_a \mid a \in X\}$ where for each $a \in X$

$$R_a = \{x \in X \mid x \geq a\}$$

(A) Prove that the right order topology is a T_0 topology in which the intersection of an arbitrary family of open sets is open.

(B) Prove that $x \leq y$ iff $x \in \overline{\{y\}}$ with respect to the right order topology on X.

(C) Conversely, given any T_0 topology $\mathfrak{J}$ on X in which the intersection of an arbitrary family of open sets is open, prove that the relation ρ on X where $x \rho y$ is defined by $x \in \overline{\{y\}}$, is an order relation with respect to which $\mathfrak{J}$ is exactly the right order topology.

5. (A) Prove that the function $F : \mathbb{R}^2 \longrightarrow$ defined by $F((x, y)) = \max(x, y)$ for every $(x, y) \in \mathbb{R}^2$ is continuous.

(A) Prove that the function $G : \mathbb{R} \longrightarrow$ defined by $G(x) = |x|$ for every $x \in \mathbb{R}^2$ is continuous.

(A) Given a topological space X, and continuous maps $f, g : X \longrightarrow \mathbb{R}$ prove that $\max(f, g)$ is continuous.

(A) Given a topological space X, and a continuous map $h : X \longrightarrow \mathbb{R}$ prove that $|h|$ is continuous.

6. Let X, Y be topological spaces, Y be Hausdorff.

(A) For any continuous functions $f, g : X \longrightarrow Y$ prove that

$$\{x \in X \mid f(x) = g(x)\}$$

is a closed set.

(B) If $f|_D = g|_D$ where D is a dense subset of X, prove that $f = g$.

7. Let X be a topological space.

(A) For any $A \subseteq X$, let the function

$$\Xi_A : X \longrightarrow \mathbb{R}$$

be defined by

$$\Xi_A(x) = \begin{cases} 1 & \text{if} \quad x \in A \\ 0 & \text{if} \quad x \notin A \end{cases}$$

for any $x \in X$. Prove that Ξ_A is continuous iff A is open and closed.

(B) For any topological space Y, every map $f : X \longrightarrow Y$ is continuous iff X has the discrete topology. (All subsets of X are open !)

(C) For any topological space Y, every map $f : Y \longrightarrow X$ is continuous iff X has the discrete topology. (No proper subset of X is open !)

8. Let X be a topological space.

(A) If $\widetilde{\mathfrak{H}}$ is the set of all open and dense subsets in X, prove that $\mathfrak{H} = \widetilde{\mathfrak{H}} \cup \{\emptyset\}$ is a topology on X.

(B) Let $\widetilde{X}$ be the topological space with carrier set X and topology $\mathfrak{H}$. Prove that a function $f : \widetilde{X} \longrightarrow \mathbb{R}$ is continuous iff it is constant.

(C) Let Y be another topological space and $f : X \longrightarrow Y$ be a continuous and open map [1]. Prove that $f : \widetilde{X} \longrightarrow \widetilde{Y}$ is continuous.

9. Let X, Y be topological spaces, Y be Hausdorff. Consider continuous maps

$$f : X \longrightarrow Y \quad \text{and} \quad g : Y \longrightarrow X$$

such that

$$g \circ f = \mathrm{Id}_X \ .$$

[1]That is, f sends open sets into open sets.

(A) Prove that X is a Hausdorff space, too.

(B) Given a continuous map $h : Y \longrightarrow Y$, prove that the set $\{y \in Y \mid h(y) = y\}$ is closed in Y.

(C) Prove that $f(X)$ is closed in Y.
(*Hint* : For any $y \in Y$, notice that $f \circ g(y) = y$ iff there exists $x \in X$ such that $y = f(x)$.)

(D) Is $f(X)$ necessarily open in Y ?

10. (A) Let X be a Hausdorff space, $h : X \longrightarrow X$ be continuous functions. Prove that

$$\mathrm{Fix}(h) = \{x \in X \mid h(x) = x\}$$

is a closed set.

(B) Given an open interval $\mathrm{I} = (a, b) \subseteq \mathbb{R}$, write down a continuous function $f : (a, b) \longrightarrow (a, b)$ such that $\mathrm{Fix}(f) = \emptyset$ and

$$\lim_{x \longrightarrow a^+} f(x) = a \ , \quad \lim_{x \longrightarrow b^-} f(x) = b$$

(C) Prove that every open subset V of $\mathbb{R}$ can be written as the union of countably many disjoint intervals.

(D) For any closed $K \subseteq \mathbb{R}$ prove that there exists a homeomorphism $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $\mathrm{Fix}(f) = K$.

11. Given a topological space X, a set $A \subseteq X$ is said to be a *retract* of X if there exists a continuous map $r : X \longrightarrow A$ such that $r|_A = \mathrm{Id}_A$.

(A) Prove that $\Delta = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ is a retract of $\mathbb{R}^2$.

(B) If A is a retract of B (relative topology) and B is a retract of X, prove that A is a retract of X.

(C) Let X, Y be topological spaces and $f : X \longrightarrow Y$ be a continuous map. Prove that the graph of f, that is, the set

$$\mathrm{graph}(f) = \{(x, f(x)) \mid x \in X\}$$

is a retract of $X \times Y$.

(D) If R is a retract of X and $A \subseteq X$, prove that $R \cap A$ is not necessarily a retract of A. (relative topology)

(E) Is a retract necessarily closed ?

12. In a topological space X be $M, N \subseteq X$ are said to be *mutually separated* if

$$\overline{M} \cap N = M \cap \overline{N} = \emptyset.$$

Let X be a topological space such that $X = A \cup B$ where $A - B$ and $B - A$ are mutually separated sets.

(A) Prove that $\overline{(B-A)} \cap A \subseteq B$

(B) Prove that for any $M \subseteq X$

$$\overline{M} \cap A \subseteq [\overline{(M \cap A)} \cap A] \cup [\overline{(M \cap B)} \cap B]$$

(C) Prove that

$$\overline{M} = \mathrm{Cl}_A(M \cap A) \cup \mathrm{Cl}_B(M \cap B)$$

(D) Prove that a set $N \subseteq X$ is closed if $N \cap A$, $N \cap B$ are closed subsets of A, B respectively.

(E) If $f : X \longrightarrow Y$ is a function such that $f|_A : A \longrightarrow Y$ and $f|_B : B \longrightarrow Y$ are continuous, prove that f is continuous.