Genaral Topology , 2008 Fall

PROBLEMS (2)

1. Let (X, d) be a metric space.

(A) Prove that $A \subseteq X$ is closed iff $\lim a_n \in A$ for each convergent sequence $a_n \in A$.

(B) Prove that $B[\,a, R\,]$ is a closed set for any $a \in X$ and any $R \geq 0$.

(C) Conclude that

$$\overline{B(a, R)} \subseteq B[\,a, R\,]$$

for any $a \in X$ and $R \geq 0$.

(D) Give an example of a metric space (X, d) and $a \in X$ and $R \geq 0$ such that

$$\overline{B(a, R)} \neq B[\,a, R\,]$$

2. Let X be a set and and $M \subseteq X$.

(A) Prove that the family $\mathfrak{H}$ consisting of the empty sets and subsets of X containing M is a topology on X.

(B) Describe the closure operator with respect to the topology $\mathfrak{H}$.

3. Let X be a set and and $N \subseteq X$.

(A) Prove that the map $\mathfrak{k} : 2^X \longrightarrow 2^X$ defined by $\mathfrak{k}\Big(\emptyset\Big) = \emptyset$ and

$$\mathfrak{k}\Big(A\Big) = A \cup N$$

for each nonempty $A \subseteq X$ satisfies the Kuratowski closure axioms.

(B) Describe the topology defined by the closure operator $\mathfrak{k}$.

4. A topological space is T_0 iff $x \notin \overline{\{y\}}$ or $y \notin \overline{\{x\}}$ for any distinct $x, y \in X$.

5. A topological space X is said to be *symmetric* if for any $x, y \in X$, $x \in \overline{\{y\}}$ holds iff $y \in \overline{\{x\}}$ holds.

(A) Prove that the Sierpinski topology is not symmetric.

(B) Prove that a T_1 topological space is symmetric.

(C) Give an example of a space which is symmetric but not T_1.

6. (A) Is the intersection of two dense subsets in a topological space always dense ?

(B) Let X be a topological space. Prove that the intersection of two open and dense subsets of X is open and dense.

7. Let X be an infinite set with the cofinite topology.

(A) Prove that $A \subseteq X$ is dense iff A is infinite.

(B) Given an infinite set Y, prove that the cofinite topology is the largest topology on Y with respect to which all infinite subsets of Y are dense.

(C) Given an infinite set Z, what is the topology on Z with respect to which the only dense subset of Z is Z itself ?

8. Let X be a topological space.

(A) For any $H \subseteq X$, prove that $a \in \overline{H}$ iff every neighbourhood of a has a nonempty intersection with H.

(B) If $V \subseteq X$ is open and $D \subseteq X$ is dense, prove that

$$\overline{V \cap D} = \overline{V}$$

(C) Prove that the following are equivalent for a topological space X:

(i) For any open $U \subseteq X$ and closed $G \subseteq X$ with $U \subseteq G$, there exists a set $A \subseteq X$ such that $A^\circ = U$ and $\overline{A} = G$.

(ii) X contains a dense subset which has empty interior. (*Hint :* Try

$$A = U \cup (G - G^\circ) \cup ((G \cap \Delta) - \overline{U})$$

where Δ is a dense set with empty interior.)

9. Let X be a topological space

(A) Give a counerexample to prove that

$$\overline{\left(\bigcup_{A \in \mathfrak{A}} A\right)} = \bigcup_{A \in \mathfrak{A}} \overline{A} \ .$$

does <u>not</u> hold for an arbitrary family $\mathfrak{A} \subseteq 2^X$ <u>in general</u>.

(B) A family $\mathfrak{M}$ of subsets of X is said to be *locally finite* if each $x \in X$ has a neighbourhood that intersects only finitely many members of $\mathfrak{M}$. Prove that the identity in (A) holds for a locally finite family.

10. A topological space is referred to as a *door space* if every subset thereof is either open or closed or both. A point $a \in X$ will be referreed to as an *accumulation point* if it is an accumulation point of the set X.

(A) Let X be a door space. If the point $a \in X$ is an accumulation point, prove that for any neighbourhood N of a the set $N - \{a\}$ is open. [1]

(B) Prove that a Hausdorff door space has at most one accumulation point. [2]

(C) Give an example of a Hausdorff door space with no accumulation point.

(D) Prove that

$$Y = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$$

with respect to the relative topology that it inherits from $\mathbb{R}$ is a Hausdorff door space with exactly one accumulation point.

(E) By (B) it is clear that

$$Z = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{2 - \frac{1}{n} \mid n \in \mathbb{N}\} \cup \{2\}$$

with respect to the relative topology that it inherits from $\mathbb{R}$ is <u>not</u> a door space. Exhibit a subset of Z which is neither open nor closed.

(F) Give an example of a door space with at least two accumulation points.

11. Let (X, d) be a metric space.

(A) If $b \in X$ is an accumulation point of $B(a, R)$ prove that $d(a, b) \leq R$.

(B) Let $c \in X$ be a point such that $d(a, c) \leq R$. Is c an accumulation point of $B(a, R)$?

12. Let X be a topological space. Let A' denote the derived set of A for any $A \subseteq X$.

(A) Prove that $\overline{A} = A \cup A'$ for any $A \subseteq X$

(B) Give an example of a topological space X and $A \subseteq X$ such that A' is not closed.

(C) Prove that in order for A' to be a closed set for any $A \subseteq X$, it is necessary that $\{a\}'$ be closed for any $a \in X$.[3] In particular, in a T_1 topological space, derived sets are closed.

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[1] Remember that a point $a \in X$ is an accumulation point (of the whole space) iff $N - \{a\} \neq \emptyset$ for any neighbourhood N of a. Equivalently, a <u>fails</u> to be an accumulation point iff $\{a\}$ is open .

[2] Consider distinct $a, b \in X$ and respective disjoint open neighbourhooods U, V thereof. Let a be an accumulation point and consider the neighbourhood $N = U \cup \{b\}$ of a.

[3] J. L. Kelley ascribes this result to C. T. Yang .