

P R O B L E M S (8)

1. (A) Let $\mathfrak{T}, \mathfrak{T}'$ be topologies on X and $\mathfrak{T} \subseteq \mathfrak{T}'$. (In other words, \mathfrak{T}' is *stronger*, or *finer* than \mathfrak{T} .) If $A \subseteq X$ is compact with respect to \mathfrak{T}' , prove that $A \subseteq X$ is also compact with respect to \mathfrak{T} .

(B) If $A \subseteq \mathbb{R}$ is compact in \mathbb{R}_{sor} , prove that $A \subseteq \mathbb{R}$ is compact in \mathbb{R} , as well.

(C) For any $a, b \in \mathbb{R}$ with $a < b$, prove that $(a, b), [a, b), (a, b], [a, b]$ are not compact in \mathbb{R}_{sor} .

(D) If $A \subseteq \mathbb{R}$ is compact in \mathbb{R}_{sor} , prove that A is bounded, A is closed in \mathbb{R}_{sor} and contains no strictly increasing sequence.

(E) If $A \subseteq \mathbb{R}$ is not compact in \mathbb{R}_{sor} , prove that A contains a strictly increasing sequence. (*Hint*: There exists a countable (why?) open cover $\mathfrak{A} = \{V_i\}_{i=1}^{\infty}$ of A with no finite subcover. Choose $x_n = \inf\left(A - \bigcup_{i=1}^n V_i\right)$.) Conclude that A is compact in \mathbb{R}_{sor} iff it is closed in \mathbb{R}_{sor} , bounded and contains no strictly increasing sequence. ¹

(F) If $A \subseteq \mathbb{R}$ is compact in \mathbb{R}_{sor} , prove that for any $a, b \in A$ with $a < b$, there exists $m \in \mathbb{R}$ with $a < m < b$ such that $A \cap (m, b) = \emptyset$.

(G) Prove that $A \subseteq \mathbb{R}$ is compact in \mathbb{R}_{sor} iff A is bounded, A is closed in \mathbb{R}_{sor} and $-A = \{x \in \mathbb{R} \mid -x \in A\}$ is discrete.

(H) Prove that a compact subset of \mathbb{R}_{sor} is countable.

¹M. S. Espelie, J. E. Joseph – Compact subsets of the Sorgenfrey line, *Mathematics Magazine* 49 (1976) 250-251