PROBLEMS (7)

1. (A) Prove that in a regular space the closure of a compact set is compact.

(B) Give an example of a topological space which has a proper compact subset whereof the closure is not compact.

(C) Give an example of a topological space a which is not regular but in which the closure of each compact set is compact.

2. Let $X = \mathbb{R}^2$ and $d : X \times X \longrightarrow \mathbb{R}$ be the function defined by

$$d(x, y) = \begin{cases} \| x - y \| & \text{if } x, y \in \mathbb{R}^2 \text{ are linearly dependent} \\ \| x \| + \| y \| & \text{if } x, y \in \mathbb{R}^2 \text{ are linearly independent} \end{cases}$$

(A) Prove that d is a metric. (Usually referred to as the *French railroad* metric. Can you guess why ?)

(B) Prove that $(a_n, b_n) \in \mathbb{R}^2$ converges to $(a, 0) \in \mathbb{R}^2$ iff a_n converges to a in the usual sense and b_n is eventually 0.

(C) Prove that with respect to the topology induced by d on it, $X = \mathbb{R}^2$ is normal but not separable.

3. A subset of a topological space is referred to as a *G_δ-set* if it is the intersection of a countable family of open sets. A topological space is said to be *perfectly normal* if it is normal and every closed subset thereof is a G_δ-set.

(A) Given a topological space X and a continuous map $f : X \longrightarrow \mathbb{R}$, prove that $f^{-1}(t)$ is a G_δ-set for every $t \in \mathbb{R}$.

(B) If Y is a normal space, prove that for every closed G_δ-set $B \subseteq Y$, there exists a continuous function $g : Y \longrightarrow \mathbb{R}$ such that $B = g^{-1}(0)$.

(C) Prove that a pseudometrisable space is perfectly normal.

(D) Is $[0, 1]^{\mathbb{R}}$ perfectly normal ? (*Hint* : Consider singletons !)