

P R O B L E M S (5)

1. (A) Let X, Y be topological spaces. Given compact $K \subseteq X$, $y \in Y$ and an open $W \subseteq X \times Y$ such that $K \times \{y\} \subseteq W$, prove that y has a neighbourhood $V \subseteq Y$ such that $K \times V \subseteq W$.

(B) If X is compact, prove that the projection $p : X \times Y \rightarrow Y$ onto the second component is a closed map.

(C) Prove that a topological space Z is compact iff the projection $p : Z \times M \rightarrow M$ onto the second component is a closed map for every topological space M .

2. ¹ Given a metric space (X, d) , let

$$V(a, r) = \{x \in X \mid d(a, x) < r\}$$

$$V[a, r] = \{x \in X \mid d(a, x) \leq r\}$$

for $a \in X$ and $r > 0$.

(A) Prove that $\overline{V(a, r)} \subseteq V[a, r]$ for any $a \in X$ and $r > 0$.

(B) For each $a \in X$, prove that the function $f : X \rightarrow \mathbb{R}$ defined by $f(x) = d(x, a)$ for $x \in X$, is continuous.

(C) Suppose that X be the union of disjoint sets $A, K \subseteq X$, where A is closed and K is compact. Prove that for each $a \in A$ there exist $r > 0$ such that

$$\overline{V(a, r)} \not\subseteq V[a, r]$$

¹M. B. Nathanson, *American Mathematical Monthly* 82 (1975) 738-741