

P R O B L E M S (4)

1. Let  $X$  be a topological space.

(A) For any  $A \subseteq X$ , let the function

$$\Xi_A : X \longrightarrow \mathbb{R}$$

be defined by

$$\Xi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

for any  $x \in X$ . Prove that  $\Xi_A$  is continuous iff  $A$  is open and closed.

(B) For any topological space  $Y$ , every map  $f : X \longrightarrow Y$  is continuous iff  $X$  has the discrete topology. (All subsets of  $X$  are open !)

(C) For any topological space  $Y$ , every map  $f : Y \longrightarrow X$  is continuous iff  $X$  has the discrete topology. (No proper subset of  $X$  is open !)

2. Given a topological space  $X$ , a set  $A \subseteq X$  is said to be a *retract* of  $X$  if there exists a continuous map  $r : X \longrightarrow A$  such that  $r|_A = \text{Id}_A$ .

(A) Let  $X, Y$  be topological spaces and  $f : X \longrightarrow Y$  be a continuous map. Prove that the graph of  $f$ , that is, the set

$$\text{graph}(f) = \{(x, f(x)) \mid x \in X\}$$

is a retract of  $X \times Y$ .

(B) Is a retract necessarily closed ?

3. (A) Prove that the product of  $T_1$  spaces is a  $T_1$  space .

(B) Let  $X, Y$  be topological spaces,  $q : X \longrightarrow Y$  be a quotient map. If  $X$  is  $T_1$ , prove that  $Y$  is a  $T_1$  space iff  $q^{-1}(\{y\})$  is closed, for every  $y \in Y$ .

(C) Give an example of a quotient map  $q : X \longrightarrow Y$  such that  $X$  is  $T_1$  whereas  $Y$  is not  $T_1$ .

S U P P L E M E N T T O P R O B L E M S (1)

1. Let  $X$  be a topological space. Given  $A \subseteq X$ , let  $\overline{A}$  denote the closure of  $A$ .

(A) For a finite family  $\mathfrak{A}$  of subsets of  $X$ , prove that

$$\overline{\cup \mathfrak{A}} = \cup \{\overline{A} \mid A \in \mathfrak{A}\} .$$

(B) Give a counterexample to show that the above identity does not hold for an arbitrary family  $\mathfrak{B}$  of subsets of  $X$ , in general.

(C) A family  $\mathfrak{M}$  of subsets of  $X$  is said to be *locally finite* if each  $x \in X$  has a neighbourhood  $V$  such that the set

$$\{M \in \mathfrak{M} \mid M \cap V \neq \emptyset\}$$

has finite cardinality. Prove that the identity in (A) holds for a locally finite family.