PROBLEMS (3)

1. Let X, Y be topological spaces, Y be Hausdorff. Consider continuous maps

$$f : X \longrightarrow Y \text{ and } g : Y \longrightarrow X$$

such that

$$g \circ f = \mathrm{Id}_X$$

Id_X being the identity map on X.

(A) Prove that X is a Hausdorff space, too.

(B) Given a continuous map $h : Y \longrightarrow Y$, prove that the set $\{y \in Y \mid h(y) = y\}$ is closed in Y.

(C) Prove that $f(X)$ is closed in Y.
(*Hint* : For any $y \in Y$, notice that $f \circ g(y) = y$ iff there exists $x \in X$ such that $y = f(x)$.)

(D) Is $f(X)$ necessarily open in Y ?

2. (A) Let X, Y be topological spaces, Y be Hausdorff, $f, g : X \longrightarrow Y$ be continuous functions. Prove that

$$\{x \in X \mid f(x) = g(x)\}$$

is a closed set.

(B) Let Z be a Hausdorff space, $h : Z \longrightarrow Z$ be continuous functions. Prove that

$$\mathrm{Fix}(h) = \{x \in X \mid h(x) = x\}$$

is a closed set.

(C) Given an open interval $\mathrm{I} = (a, b) \subseteq \mathbb{R}$, write down a continuous function $f : (a, b) \longrightarrow (a, b)$ such that $\mathrm{Fix}(f) = \emptyset$ and

$$\lim_{x \longrightarrow a^+} f(x) = a \ , \quad \lim_{x \longrightarrow b^-} f(x) = b$$

(D) Prove that every open subset V of $\mathbb{R}$ can be written as the union of countably many disjoint intervals.

(E) For any closed $K \subseteq \mathbb{R}$ prove that there exists a homeomorphism $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $\mathrm{Fix}(f) = K$.