PROBLEMS (1)

1. (A) Let F be a finite set. Prove that there exists only one Hausdorff topology on F.

(B) Given a set M, prove that the cofinite topology on M is Hausdorff iff M is finite.

(C) Let X be an infinite set with the cofinite topology. Prove that $A \subseteq X$ is dense iff A is infinite.

(D) Given an infinite set Y, prove that the cofinite topology is the largest topology on Y with respect to which all infinite subsets of Y are dense.

(E) Given an infinite set Z, what is the topology on Z with respect to which the only dense subset of Z is Z itself ?

2. Let X be a topological space.

(A) For any $H \subseteq X$, prove that $a \in \overline{H}$ iff every neighbourhood of a has a nonempty intersection with H.

(B) If $V \subseteq X$ is open and $D \subseteq X$ is dense, prove that

$$\overline{V \cap D} = \overline{V}$$

(C) Prove that the following are equivalent for a topological space X:
(i) For any open $U \subseteq X$ and closed $G \subseteq X$ with $U \subseteq G$, there exists a set $A \subseteq X$ such that $A^\circ = U$ and $\overline{A} = G$.
(ii) X contains a dense subset which has empty interior. (*Hint :* Try

$$A = U \cup (G - G^\circ) \cup ((G \cap \Delta) - \overline{U})$$

where Δ is a dense set with empty interior.)

3. A topological space is called a *door space* if every subset thereof is either open or closed or both.

(A) In a topological space X, prove that a point $a \in X$ fails to be an accumulation point iff the set $\{a\}$ is open.

(B) Let Y be a door space. If the point $b \in Y$ is an accumulation point, prove that for any neighbourhood N of b the set $N - \{b\}$ is open.

(C) Prove that a Hausdorff door space has at most one accumulation point.

(D) Give examples of (1) a Hausdorff door space with no accumulation point, (2) a Hausdorff door space with exactly one accumulation point, (3) a door space with at least two accumulation points.