Genaral Topology

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FINAL EXAMINATION

January 2008

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1. A topological space is referred to as a *door space* if every subset thereof is either open or closed or both.

(A) Let X be a door space. If the point $a \in X$ is an accumulation point, prove that for any neighbourhood N of a the set $N - \{a\}$ is open. [1]

(B) Prove that a Hausdorff door space has at most one accumulation point. [2]

(C) Give an example of a Hausdorff door space with no accumulation point. Prove that

$$Y = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$$

is a door space with respect to the relative topology that it inherits from $\mathbb{R}$.

(D) By (B) it is clear that

$$Z = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{2 - \frac{1}{n} \mid n \in \mathbb{N}\} \cup \{2\}$$

with respect to the relative topology that it inherits from $\mathbb{R}$ is not a door space. Exhibit a subset of Z which is neither open nor closed.

(E) Give an example of a door space with at least two accumulation points.

2. A subset of a topological space is referred to as a *G_δ-set* if it is the intersection of a countable family of open sets. A topological space is said to be *perfectly normal* if it is normal and every closed subset thereof is a G_δ-set.

(A) Given a topological space X and a continuous map $f : X \longrightarrow \mathbb{R}$, prove that $f^{-1}(t)$ is a G_δ-set for every $t \in \mathbb{R}$.

(B) If Y is a normal space, prove that for every closed G_δ-set $B \subseteq Y$, there exists a continuous function $g : Y \longrightarrow \mathbb{R}$ such that $B = g^{-1}(0)$.

[1] Remember that a point $a \in X$ is an accumulation point (of the whole space) iff $N - \{a\} \neq \emptyset$ for any neighbourhood N of a. Equivalently a fails to be an accumulation point iff $\{a\}$ is open .

[2] Consider distinct $a, b \in X$ and respective disjoint open neighbourhooods U, V thereof. Let a be an accumulation point and consider the neighbourhood $N = U \cup \{b\}$ of a.

(C) Prove that a pseudometrisable space is perfectly normal.

(D) Is $[0,1]^{\mathbb{R}}$ perfectly normal ? [3]

3. Given a metric space (X,d), $r>0$ and any $A \subseteq X$, let $V_r(A)$ be defined by

$$V_r(A) = \bigcup_{a\in A} V_r(a)$$

where, as usual

$$V_r(x) = \{y \in X \mid d(x,y) < r\}$$

for any $x \in X$.

Suppose that the metric space (X,d) is bounded in the sense that $X = V_R(x_0)$ for some $x_0 \in X$ and $R > 0$. Let $\tilde{X}$ stand for the set of closed subsets of X. Consider the function $\tilde{d} : \tilde{X} \times \tilde{X} \longrightarrow \mathbb{R}$ defined by

$$\tilde{d}(K,L) = \inf\{r > 0 \mid K \subseteq V_r(L), L \subseteq V_r(K)\}.$$

for $K, L \in \tilde{X}$.

(A) Prove that $\tilde{d}$ is a metric on $\tilde{X}$.

(B) Let $Y = \mathbb{R}_{>0} = \{t \in \mathbb{R} \mid t > 0\}$. Consider $d_1,\, d_2 : Y \times Y \longrightarrow \mathbb{R}$ defined by

$$d_1(s,t) = |\frac{s}{1+s} - \frac{t}{1+t}| \ ,$$

$$d_2(s,t) = \min\{1, |s-t|\}$$

for $s,t \in \mathbb{R}$. Prove that d_1, d_2 are metrics which induce the usual topology on Y.

(C) Consider $\tilde{d}_1, \tilde{d}_2$ on $\tilde{Y}$. Let $\tilde{y}_n = \{1,2,3,\ldots,n\}$ for each $n \in \mathbb{N}$. Prove that $\lim_{n\longrightarrow\infty} \tilde{y}_n = \mathbb{N}$ with respect to $\tilde{d}_1$.

(D) Do $\tilde{d}_1$ and $\tilde{d}_2$ induce the same topology on $\tilde{Y}$?

4. Let θ be an irrational number and $Z = \{(x,y) \in \mathbb{Q}^2 : y \geq 0\}$. For each $(x,y) \in Z$ and $\varepsilon > 0$ let $V_\varepsilon((x,y))$ be the set

$$\{(x,y)\} \cup \{(x',0) \in \mathbb{Q}^2 : |x' - (x - \theta y)| < \varepsilon\} \cup \{(x'',0) \in \mathbb{Q}^2 : |x'' - (x + \theta y)| < \varepsilon\}$$

(A) Prove that Z can be topologised so that for each $(x,y) \in Z$, the family $\{V_\varepsilon((x,y))\}_{\varepsilon>0}$ constitutes a neighbourhood base for (x,y).

(B) Let Z be topologised as above. Prove that Z is a Hausdorff space.

(C) Given any distinct (x,y), $(x',y') \in Z$ and any closed neighbourhoods M, N of (x,y), (x',y') respectively, prove that $M \cap N \neq \emptyset$.

(D) Prove that Z is connected.

[3] Consider singletons !

5. In a topological space X be $M, N \subseteq X$ are said to be *mutually separated* if

$$\overline{M} \cap N = M \cap \overline{N} = \emptyset.$$

Let X be a topological space such that $X = A \cup B$ where $A - B$ and $B - A$ are mutually separated sets.

(A) Prove that $\overline{(B - A)} \cap A \subseteq B$

(B) Prove that for any $M \subseteq X$

$$\overline{M} \cap A \subseteq [\overline{(M \cap A)} \cap A] \cup [\overline{(M \cap B)} \cap B]$$

(C) Prove that

$$\overline{M} = \mathrm{Cl}_A(M \cap A) \cup \mathrm{Cl}_B(M \cap B)$$

(D) Prove that a set $N \subseteq X$ is closed if $N \cap A$, $N \cap B$ are closed subsets of A, B respectively.

(E) If $f : X \longrightarrow Y$ is a function such that $f|_A : A \longrightarrow Y$ and $f|_B : B \longrightarrow Y$ are continuous, prove that f is continuous.

6. Let $X = \mathbb{Z} \times \mathbb{Z}$ and let τ consist of sets $A \subseteq X$ with the property that either $(0,0) \notin A$ or there exists $N \subseteq A$ with $(0,0) \in N$ such that $\Big(\{m\} \times \mathbb{Z}\Big) \cap \Big(X - N\Big)$ is finite for all except finitely many $m \in \mathbb{Z}$.

(A) Prove that τ constitutes a topology on X. [4]

(B) Prove that X [5] is Hausdorff and regular . Conclude that X is in fact T_4 . [6]

(C) Prove that $(0,0) \in X$ has no compact neighbourhood. Conclude that $X - \{(0,0)\}$ is dense.

(D) Prove that every neighbourhood of $(0,0) \in X$ is closed.

(E) Prove that no sequence in $X - \{(0,0)\}$ converges to $(0,0)$.

(F) Let the sequence $(y_n)_{n \in \mathbb{N}}$ be an enumeration of $X - \{(0,0)\}$. In other words, let

$$\{y_n \mid n \in \mathbb{N}\} = X - \{(0,0)\}.$$

Prove that the sequence has a subnet that converges to $(0,0)$. Conclude that a subnet of a sequence is not necessarily a subsequence.

[4] The system (X, τ) is the so called *Arens-Fort Space* . See : *Counterexamples in Topology* by L. A. Steen, J. A. Seebach, jr.

[5] X henceforth designates the topological space (X, τ).

[6] Being countable, X is trivially Lindelöf.