SEVENTH LECTURE

Lie Brackets

Given vector fields $A, B \in \mathfrak{X}(M)$, the **Lie bracket** of A and B is the vector field $[A,B] \in \mathfrak{X}(M)$ defined for each $m \in M$ by

$$[A,B]|_m = \lim_{t \longrightarrow 0} \frac{1}{t} \left\{ (T_m\varphi_t)^{-1}(B_{\varphi(m,t)}) - B_m \right\}$$

equivalently by

$$[A,B]|_m = \left.\frac{d}{dt}\right|_{t=0} (T_m\varphi_t)^{-1}(B_{\varphi(m,t)})$$

where φ is a local flow on a neighbourhood of m generated by the vector field A.

EXAMPLE : Let (x,y) be the identity chart on $\mathbb{R}^2$ and consider

$$A = \frac{\partial}{\partial x} \quad , \quad B = -y\,\frac{\partial}{\partial x} + x\,\frac{\partial}{\partial y} \in \mathfrak{X}(\mathbb{R}^2)$$

Since A is the velocity field of the flow φ defined by

$$\varphi((x,y),t) = (x+t,y)$$

and

$$T_{(x,y)}\varphi_t \left(\left.\frac{\partial}{\partial x}\right|_{(x,y)} \right) = \left.\frac{\partial}{\partial x}\right|_{(x+t,y)} \quad , \quad T_{(x,y)}\varphi_t \left(\left.\frac{\partial}{\partial y}\right|_{(x,y)} \right) = \left.\frac{\partial}{\partial y}\right|_{(x+t,y)}$$

we easily conclude that

$$[A,B]|_{(x,y)} = \lim_{t \longrightarrow 0} \frac{1}{t} \left\{ (T_{(x,y)}\varphi_t)^{-1}(B_{(x+t,y)}) - B_{(x,y)} \right\}$$

$$= \lim_{t \longrightarrow 0} \frac{1}{t} \left\{ (T_{(x,y)}\varphi_t)^{-1} \left(-y\,\left.\frac{\partial}{\partial x}\right|_{(x+t,y)} + (x+t)\,\left.\frac{\partial}{\partial y}\right|_{(x+t,y)} \right) - \left(-y\,\left.\frac{\partial}{\partial x}\right|_{(x,y)} + x\,\left.\frac{\partial}{\partial y}\right|_{(x,y)} \right) \right\}$$

$$= \lim_{t \longrightarrow 0} \frac{1}{t} \left\{ \left(-y\,\left.\frac{\partial}{\partial x}\right|_{(x,y)} + (x+t)\,\left.\frac{\partial}{\partial y}\right|_{(x,y)} \right) - \left(-y\,\left.\frac{\partial}{\partial x}\right|_{(x,y)} + x\,\left.\frac{\partial}{\partial y}\right|_{(x,y)} \right) \right\}$$

$$= \lim_{t \longrightarrow 0} \frac{1}{t} \{ t \left. \frac{\partial}{\partial y} \right|_{(x,y)} \} = \left. \frac{\partial}{\partial y} \right|_{(x,y)}$$

On the other hand, since B is the velocity field of the flow ψ defined by

$$\psi((x,y),t) = (x(t), y(t))$$

where we have written $x(t) = x \cos t - y \sin t$ and $y(t) = x \sin t + y \cos t$ for brevity, we have

$$T_{(x,y)} \psi_t \left(\left. \frac{\partial}{\partial x} \right|_{(x,y)} \right) = \cos t \left. \frac{\partial}{\partial x} \right|_{(x(t),y(t))} + \sin t \left. \frac{\partial}{\partial y} \right|_{(x(t),y(t))}$$

$$T_{(x,y)} \psi_t \left(\left. \frac{\partial}{\partial y} \right|_{(x,y)} \right) = -\sin t \left. \frac{\partial}{\partial x} \right|_{(x(t),y(t))} + \cos t \left. \frac{\partial}{\partial y} \right|_{(x(t),y(t))}$$

and we obtain

$$[B, A]|_{(x,y)} = \lim_{t \longrightarrow 0} \frac{1}{t} \left\{ (T_{(x,y)} \psi_t)^{-1} (A_{(x+t,y)}) - A_{(x,y)} \right\}$$

$$= \lim_{t \longrightarrow 0} \frac{1}{t} \left\{ (T_{(x,y)} \psi_t)^{-1} \left(\left. \frac{\partial}{\partial x} \right|_{(x(t),y(t))} \right) - \left. \frac{\partial}{\partial x} \right|_{(x,y)} \right\}$$

$$= \lim_{t \longrightarrow 0} \frac{1}{t} \left\{ \left(\cos t \left. \frac{\partial}{\partial x} \right|_{(x,y)} - \sin t \left. \frac{\partial}{\partial y} \right|_{(x,y)} \right) - \left. \frac{\partial}{\partial x} \right|_{(x,y)} \right\}$$

$$= \lim_{t \longrightarrow 0} \left\{ \frac{\cos t - 1}{t} \left. \frac{\partial}{\partial x} \right|_{(x,y)} - \frac{\sin t}{t} \left. \frac{\partial}{\partial y} \right|_{(x,y)} \right\} = - \left. \frac{\partial}{\partial y} \right|_{(x,y)}$$

Since a vector field is entirely determined by its action on scalar fields the following is of immense theoretical and practical value :

THEOREM : For any vector fields $A, B \in \mathfrak{X}(M)$, and any scalar field $f \in \mathfrak{F}(M)$,

$$[A, B]f = A(Bf) - B(Af)$$

Proof : For any $m \in M$,

$$[A, B]f(m) = \left\{ \lim_{t \longrightarrow 0} \frac{1}{t} \left[(T_m \varphi_t)^{-1} (B_{\varphi(m,t)}) - B_m \right] \right\} f$$

$$= \lim_{t \longrightarrow 0} \frac{1}{t} \left\{ \left[(T_m \varphi_t)^{-1} (B_{\varphi(m,t)}) \right] f - B_m f \right\}$$

$$= \lim_{t \longrightarrow 0} \frac{1}{t} \left\{ \left[T_{\varphi(m,t)} \varphi_{-t} (B_{\varphi(m,t)}) \right] f - B_m f \right\}$$

$$= \lim_{t \longrightarrow 0} \frac{B_{\varphi(m,t)}(f \circ \varphi_{-t}) - B_m f}{t}$$

$$= \lim_{t \longrightarrow 0} \left[\frac{B_{\varphi(m,t)}(f \circ \varphi_{-t}) - B_{\varphi(m,t)} f}{t} + \frac{B_{\varphi(m,t)} f - B_m f}{t} \right]$$

$$= \lim_{t \longrightarrow 0} \left[B_{\varphi(m,t)} \frac{f \circ \varphi_{-t} - f}{t} + \frac{Bf(\varphi(m,t)) - Bf(m)}{t} \right]$$

$$= B(-A)f(m) + ABf(m)$$

∎

A simple inspection of the action on scalar allows us to obtain the following very important corollary, whereof the second part is referred to as the **Jacobi Identity** :

COROLLARY : For any vector fields $X, Y, Z \in \mathfrak{X}(M)$,

$$[X, Y] = -[Y, X]$$

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

There is yet another routine but important application of the above theorem :

COROLLARY : For any vector fields $X, Y \in \mathfrak{X}(M)$, and any scalar fields $f, g \in \mathfrak{F}(M)$

$$[X, fY] = f[X, Y] + XfY$$

$$[gX, Y] = g[X, Y] - YgX$$

PROBLEMS

I. Compute $[A, B] \in \mathfrak{X}(\mathbb{R}^2)$ where

$$A = \frac{\partial}{\partial x} + e^{(x^2-y^3)}\frac{\partial}{\partial y}$$
$$B = \sin{(x^2+y^2)}\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$$

II. Let $A, B \in \mathfrak{X}(\mathbb{S}^2)$ be the vector fields induced by the respective flows

$$\varphi, \psi : \mathbb{S}^2 \times \mathbb{R} \longrightarrow \mathbf{S}^2$$

defined by

$$\varphi((x,y,z),t) = \left(\frac{2xe^t}{1+z+(1-z)e^{2t}} \ , \ \frac{2ye^t}{1+z+(1-z)e^{2t}} \ , \ \frac{1+z-(1-z)e^{2t}}{1+z+(1-z)e^{2t}} \right)$$
$$\psi((x,y,z),t) = (x, y\cos t - z\sin t, y\sin t + z\cos t) \quad .$$

Compute $[A, B]$ in the chart on $\mathbb{S}^2$ obtained by stereographic projection from the south pole.

III. Let $A, B \in \mathfrak{X}(\mathbb{R}P^2)$ be the vector fields induced by the respective flows

$$\varphi, \psi : \mathbb{R}P^2 \times \mathbb{R} \longrightarrow \mathbb{R}P^2$$

defined by

$$\varphi([x,y,z],t) = [x, e^t y, e^{-t} z] \quad .$$
$$\psi([x,y,z],t) = [e^t y + e^{-t} z, y, e^t z + e^{-t} x] \quad .$$

Compute $[A, B]$ in the standard charts on $\mathbb{R}P^2$.

IV. Ideally, the position and so to speak the disposition of a car can be modelled by a unit vector $\mathbf{u}$ positioned at $(x, y) \in \mathbb{R}^2$ making an angle φ with the x-axis and the perpendicular bisector of the “front wheel axle” making an angle θ with $\mathbf{u}$. For this reason the manifold

$$M = \mathbb{R}^2 \times \left(\frac{\mathbb{R}}{2\pi\mathbb{Z}}\right)^2$$

is sometimes referred to as the **car manifold .** Let us identify M with $\mathbb{R}^4$ locally and make use of the coordinates (x, y, θ, φ) with their now obvious meanings.

Consider the following vector fields on M with names inspired by the circumstances from which we have observed M to arise as a model.

$$\mathrm{St} = \frac{\partial}{\partial \theta} \qquad \text{the } \textbf{steer},$$

(fix everything, turn the driving wheel,)

$$\mathrm{Ro} = \frac{\partial}{\partial \varphi} \qquad \text{the } \textbf{rotate},$$

$$\mathrm{Dr} = \cos{(\theta+\varphi)}\frac{\partial}{\partial x} + \sin{(\theta+\varphi)}\frac{\partial}{\partial y} + \sin\theta\frac{\partial}{\partial \varphi} \qquad \text{the } \textbf{drive},$$

(fix θ, move uniformly with unit speed in the direction the front wheels point in,)

$$\mathrm{Wr} = -\sin{(\theta+\varphi)}\frac{\partial}{\partial x} + \cos{(\theta+\varphi)}\frac{\partial}{\partial y} + \cos\varphi\frac{\partial}{\partial \varphi} \qquad \text{the } \textbf{wriggle},$$

$$\mathrm{Sl} = -\sin\varphi\frac{\partial}{\partial x} + \cos\varphi\frac{\partial}{\partial y} \qquad \text{the } \textbf{slide}.$$

(A) Prove that

$$[\mathrm{Sl}, \mathrm{St}] = [\mathrm{Sl}, \mathrm{Dr}] = [\mathrm{Sl}, \mathrm{Wr}] = 0$$

$$[\mathrm{St}, \mathrm{Dr}] = \mathrm{Wr}$$
$$[\mathrm{Dr}, \mathrm{Wr}] = -\mathrm{Sl}$$
$$[\mathrm{Wr}, \mathrm{St}] = \mathrm{Dr} \quad .$$

(B) Compute the Lie brackets with Ro .