THIRD LECTURE

Tangent Vectors and Covectors

Given a differentiable manifold M, let $\hat{T}_m M$ stand for the set of differentiable maps $\lambda : \mathrm{dom}(\lambda) \longrightarrow M$ where $0 \in \mathrm{dom}(\lambda) \subseteq_{op} \mathbb{R}$ and $\lambda(0) = m$. For $\lambda, \mu \in \hat{T}_m M$ write $\lambda \sim \mu$ if

$$\left.\frac{d}{dt}\right|_{t=0} f(\lambda(t)) = \left.\frac{d}{dt}\right|_{t=0} f(\mu(t))$$

for every differentiable real valued function f defined near m. It is obvious that $\sim$ is an equivalence relation on $\hat{T}_m M$. A **tangent vector** at m (henceforth briefly a **vector** at m) is an $\sim$-equivalence class on $\hat{T}_m M$. The set of tangent vectors at $m \in M$ will be denoted by $T_m M$ and will be referred to as the **tangent space** at m.

In the sequel we shall denote the $\sim$-equivalence class containing $\lambda \in \hat{T}_m M$ by $[\lambda]$. Given a vector u at m, we shall occasionally stress that it is a vector at the point m by writing it in the form u_m or even $\mathsf{u}|_m$.

Remember that given $\mathsf{u} = [\lambda] \in T_m M$ and any differentiable f defined near m, the number

$$\left.\frac{d}{dt}\right|_{t=0} f(\lambda(t))$$

is, by its very definition, independent of the choice of λ and depends solely on u and f. This quantity is denoted by $\mathsf{u}f$ or $\mathsf{u}_m f$ or $\mathsf{u}|_m f$ and is referred to as the **derivative** of f **in the direction** u .

In the presence of a chart $x = (x^i)_{1 \le i \le n}$ with $m \in \mathrm{dom}(x)$, we consider $\xi_k \in \hat{T}_m M$ defined near $0 \in \mathbb{R}$ by $\xi_k(t) = x^{-1}(x(m) + t\mathsf{e}_k)$, for $1 \le k \le n$. The vectors $[\xi_k]$ will be of paramount importance and will be denoted by

$$\left.\frac{\partial}{\partial x^k}\right|_m .$$

Given differentiable, real valued f defined near m, the derivative

$$\left.\frac{\partial}{\partial x^k}\right|_m f$$

will be frequently used and be accorded much notational flexibility :

$$\left.\frac{\partial f}{\partial x^k}\right|_m \ , \ \frac{\partial f}{\partial x^k}(m) \ , \ \left.\frac{\partial}{\partial x^k} f\right|_m$$

It is interesting to notice that when one writes f as a function of the functions $x^1, x^2, \cdots, x^n$ in the spirit of the note on convention in the second lecture, this derivative reduces to the partial derivative of elementary calculus. Indeed

$$\left.\frac{\partial}{\partial x^k}\right|_m f = \left.\frac{d}{dt}\right|_{t=0} f(\xi_k(t)) = \left.\frac{d}{dt}\right|_{t=0} f \circ x^{-1}(x(m) + t\mathsf{e}_k) = \frac{\partial (f \circ x^{-1})}{\partial x^k}(x(m))$$

We note in particular that

$$\frac{\partial x^j}{\partial x^k}(m) = \delta^j_k \ .$$

EXAMPLE : Consider $f : \mathbb{R}P^2 \longrightarrow \mathbb{R}$ defined by

$$f([u,v,w]) = \exp\left(\frac{|u|}{3|u| + |v| + |w|}\right)$$

which is well defined and continuous on $\mathbb{R}P^2$ but differentiable only on the open set $A = \{[u,v,w] \mid u \neq 0, v \neq 0, w \neq 0\}$. Let $m = [1,2,-1]$. Using the standard chart $x = (x^1, x^2)$ we find $x^1(m) = 2$, $x^2(m) = -1$ and

$$f = \exp\left(\frac{1}{3 + x^1 - x^2}\right)$$

near m and finally

$$\left.\frac{\partial}{\partial x^2}\right|_m f = \left.\frac{\partial}{\partial x^2} \exp\left(\frac{1}{3 + x^1 - x^2}\right)\right|_{x^1=2, x^2=-1}$$

$$= -\left.\frac{1}{(3 + x^1 - x^2)^2} \exp\left(\frac{1}{3 + x^1 - x^2}\right)\right|_{x^1=2, x^2=-1} = \frac{1}{36} e^{1/6}$$

REMARK : By its very definition it is clear that a tangent vector is uniquely determined by its action on real valued differentiable functions as direction of differentiation. To be precise, given $\mathsf{u}, \mathsf{v} \in T_mM$, if $\mathsf{u}f = \mathsf{v}f$ for every differentiable, real valued f defined near m, then $\mathsf{u} = \mathsf{v}$.

Given $m \in M$, let $c : \mathbb{R} \longrightarrow M$ be the constant map with constant value m. That is $c(t) = m$ for all $t \in \mathbb{R}$. We shall call $[c] \in T_mM$ the **zero tangent vector** at m (henceforth briefly the **zero vector** at m) and denote it by $\mathbf{0}$ or $\mathbf{0}_m$. Clearly

$$\mathbf{0}f = 0$$

for any differentiable, real valued f defined near m .

Given $a \in \mathbb{R}$ and $\mathsf{u} = [\lambda] \in T_mM$, if we put $\lambda_a(t) = \lambda(at)$ for all t sufficiently close to $0 \in \mathbb{R}$, we find that for any differentiable, real valued f defined near m ,

$$[\lambda_a]f = \left.\frac{d}{dt}\right|_{t=0} f(\lambda_a(t)) = \left.\frac{d}{dt}\right|_{t=0} f(\lambda(at)) = a \left.\frac{d}{dt}\right|_{t=0} f(\lambda(t)) = a(\mathsf{u}f)$$

which depends only on $\mathsf{u} = [\lambda]$ and is independent of the choice of $\lambda \in \hat{T}_m M$ in view of the above remark. Consequently we define the product of $a \cdot_m \mathsf{u}$ (which we immediately abbreviate into $a\mathsf{u}$!) of $\mathsf{u} = [\lambda] \in T_m M$ with $a \in \mathbb{R}$ to be $[\lambda_a]$ and notice that

$$(a\mathsf{u})f = a(\mathsf{u}f)$$

for any differentiable f defined near m.

Given $\mathsf{u} = [\lambda]$, $\mathsf{v} = [\mu] \in T_m M$, and a chart $x = (x^i)_{1 \le i \le n}$ with $m \in \mathrm{dom}(x)$, if we define $\nu = \nu_{\lambda,\mu}$ by

$$\begin{aligned}\nu(t) &= x^{-1}(x(m) + (x\lambda(t) - x(m)) + (x\mu(t) - x(m))) \\ &= x^{-1}(x\lambda(t) + x\mu(t) - x(m))\end{aligned}$$

for all t sufficiently close to $0 \in \mathbb{R}$, we find that for any differentiable, real valued f defined near m ,

$$\begin{aligned}[\nu]f = \left.\frac{d}{dt}\right|_{t=0} f(\nu(t)) &= \left.\frac{d}{dt}\right|_{t=0} f \circ x^{-1}(x\nu(t)) \\ &= D_{x(m)} f \circ x^{-1} \left(\left.\frac{d}{dt}\right|_{t=0} x\nu(t) \right) \\ &= D_{x(m)} f \circ x^{-1} \left(\left.\frac{d}{dt}\right|_{t=0} (x\lambda(t) + x\mu(t) - x(m)) \right) \\ &= D_{x(m)} f \circ x^{-1} \left(\left.\frac{d}{dt}\right|_{t=0} x\lambda(t) + \left.\frac{d}{dt}\right|_{t=0} x\mu(t)) \right) \\ &= D_{x(m)} f \circ x^{-1} \left(\left.\frac{d}{dt}\right|_{t=0} x\lambda(t) \right) + D_{x(m)} f \circ x^{-1} \left(\left.\frac{d}{dt}\right|_{t=0} x\mu(t) \right) \\ &= \left.\frac{d}{dt}\right|_{t=0} f \circ x^{-1}(x\lambda(t)) + \left.\frac{d}{dt}\right|_{t=0} f \circ x^{-1}(x\mu(t)) \\ &= \mathsf{u}f + \mathsf{v}f\end{aligned}$$

which depends only on $\mathsf{u} = [\lambda]$, $\mathsf{v} = [\mu]$ and is independent of the choice of $\lambda, \mu \in \hat{T}_m M$ in view of the above remark. Consequently we define the sum $\mathsf{u} +_m \mathsf{v}$ (which we immediately abbreviate into $\mathsf{u} + \mathsf{v}$!) of $\mathsf{u} = [\lambda] \in T_m M$ and $\mathsf{v} = [\mu] \in T_m M$ to be $[\nu_{\lambda,\mu}]$ and notice that

$$(\mathsf{u} + \mathsf{v})f = \mathsf{u}f + \mathsf{v}f$$

for any differentiable f defined near m.

THEOREM : For each $m \in M$, the system $(T_m M, \mathbf{0}_m, +_m, \cdot_m)$ constitutes a vector space.

Proof : That $T_m M$ is a commutative group with respect to $+_m$ that admits $\mathbf{0}_m$ as neutral element can be seen immediately by virtue of the above remark from the following simple observations

$$(\mathsf{u} + \mathbf{0})f = \mathsf{u}f + \mathbf{0}f = \mathsf{u}f + 0 = \mathsf{u}f$$

$$(\mathsf{u} + \mathsf{v})f = \mathsf{u}f + \mathsf{v}f = (\mathsf{v} + \mathsf{u})f$$

$$(\mathsf{u} + (\mathsf{v} + \mathsf{w}))f = \mathsf{u}f + (\mathsf{v} + \mathsf{w})f = \mathsf{u}f + \mathsf{v}f + \mathsf{w}f = ((\mathsf{u} + \mathsf{v}) + \mathsf{w}))f$$

$$(\mathsf{u} + (-1)\mathsf{u})f = \mathsf{u}f - \mathsf{u}f = 0 = \mathbf{0}f$$

for $\mathsf{u}, \mathsf{v}, \mathsf{w} \in T_m M$ which are valid for any differentiable f defined near m. That this commutative group, augmented with the multiplication with elements of $\mathbb{R}$ obeys the rules

$$1\mathsf{u} = \mathsf{u}$$

$$a(b\mathsf{u}) = (ab)\mathsf{u}$$
$$a(\mathsf{u} + \mathsf{v}) = a\mathsf{u} + a\mathsf{v}$$
$$(a + b)\mathsf{u} = a\mathsf{u} + b\mathsf{u}$$

can be easily seen from the equations

$$(1\mathsf{u})f = 1(\mathsf{u}f) = \mathsf{u}f$$

$$(a(b\mathsf{u}))f = a((b\mathsf{u})f) = a(b(\mathsf{u}f)) = (ab)(\mathsf{u}f) = ((ab)\mathsf{u}))f$$
$$(a(\mathsf{u} + \mathsf{v}))f = a((\mathsf{u} + \mathsf{v})f) = a(\mathsf{u}f + \mathsf{v}f) = a(\mathsf{u}f) + a(\mathsf{v}f) = (a\mathsf{u})f + (a\mathsf{v})f = (a\mathsf{u} + a\mathsf{v})f$$
$$((a + b)\mathsf{u})f = (a + b)(\mathsf{u}f) = a(\mathsf{u}f) + b(\mathsf{u}f) = (a\mathsf{u})f + (b\mathsf{u})f = (a\mathsf{u} + b\mathsf{u})f$$

for $\mathsf{u}, \mathsf{v} \in T_m M$ which are valid for any differentiable f defined near m. ∎

REMARK : By abuse of language we shall refer to the vector space $(T_m M, \mathbf{0}_m, +_m, \cdot_m)$ as the vector space $T_m M$.

THEOREM : For each $m \in M$, and each chart $x = (x^i)_{1 \leq i \leq n}$ with $m \in \mathrm{dom}(x)$ the set

$$\mathfrak{B}_{x,m} = \left\{ \left. \frac{\partial}{\partial x^i} \right|_m \right\}_{1 \leq i \leq n}$$

constitutes a basis for $T_m M$. Given charts x, y with $m \in \mathrm{dom}(x) \cap \mathrm{dom}(y)$, the elements of the bases $\mathfrak{B}_{x,m}$ and $\mathfrak{B}_{y,m}$ are related to one another by

$$\left. \frac{\partial}{\partial y^p} \right|_m = \sum_{i=1}^{n} \frac{\partial x^i}{\partial y^p}(m) \left. \frac{\partial}{\partial x^i} \right|_m$$

Proof : Given $\mathsf{u} = [\lambda] \in T_m M$, let

$$\left. \frac{d}{dt} \right|_{t=0} x^i(\lambda(t)) = a^i \quad .$$

Since for any differentiable f defined near m we have

$$\begin{aligned} \mathsf{u}f = \left. \frac{d}{dt} \right|_{t=0} f(\lambda(t)) &= \left. \frac{d}{dt} \right|_{t=0} f \circ x^{-1}(x\lambda(t)) \\ &= \sum_{i=1}^{n} \Big(\left. \frac{d}{dt} \right|_{t=0} x^i(\lambda(t)) \Big) \frac{\partial (f \circ x^{-1})}{\partial x^i}(x(m)) \end{aligned}$$

from which we conclude that

$$\mathsf{u}f = \sum_{i=1}^{n} a^i \frac{\partial f}{\partial x^i}(m)$$

for any differentiable f defined near m hence

$$\mathsf{u} = \sum_{i=1}^{n} a^i \left.\frac{\partial}{\partial x^i}\right|_m$$

where

$$a^i = \left.\frac{d}{dt}\right|_{t=0} x^i(\lambda(t))$$

for $1 \le i \le n$. This being true for arbitrary $\mathsf{u} \in T_mM$ we infer that $\mathfrak{B}_{x,m}$ spans T_mM. It remains to see that $\mathfrak{B}_{x,m}$ is linearly independent : If

$$\sum_{k=1}^{n} b^k \left.\frac{\partial}{\partial x^k}\right|_m = \mathbf{0} \in T_mM$$

then,

$$0 = \sum_{k=1}^{n} \left.\left(b^k \frac{\partial}{\partial x^k}\right)\right|_m x^i = \sum_{k=1}^{n} b^k \delta^i_k = b^i$$

for all $1 \le i \le n$. Therefore $\mathfrak{B}_{x,m}$ is a basis for T_mM. Let's now consider $\mathfrak{B}_{y,m}$: Suppose

$$\left.\frac{\partial}{\partial y^k}\right|_m = [\eta_k]$$

where

$$\eta_k(t) = y^{-1}(y(m) + t\mathsf{e}_k)$$

for $1 \le k \le n$. For any differentiable f defined near m

$$\begin{aligned} \left.\frac{\partial}{\partial y^p}\right|_m f = \frac{\partial (f \circ y^{-1})}{\partial y^p}(y(m)) &= \sum_{i=1}^{n} \frac{\partial (x^i \circ y^{-1})}{\partial y^p}(y(m)) \frac{\partial (f \circ x^{-1})}{\partial x^i}(x(m)) \\ &= \left(\sum_{i=1}^{n} \frac{\partial x^i}{\partial y^p}(m) \left.\frac{\partial}{\partial x^i}\right|_m\right) f \end{aligned}$$

from which follows the announced relation. ∎

EXAMPLE : Consider $\mathbb{R}P^2$ and the standard charts $x = (x^1, x^2)$ and $y = (y^1, y^2)$ thereon. Let $m = [1, 2, -1] \in \mathbb{R}P^2$. Clearly $y^1(m) = -1/2$, $y^2(m) = 1/2$

$$x^1 = \frac{1}{y^2} \quad , \quad x^2 = \frac{y^1}{y^2}$$

we compute

$$\left.\frac{\partial x^1}{\partial y^1}\right|_m = 0 \quad , \quad \left.\frac{\partial x^1}{\partial y^2}\right|_m = -\left.\frac{1}{(y^2)^2}\right|_{y^1=-1/2, y^2=1/2} = -4$$

$$\left.\frac{\partial x^2}{\partial y^1}\right|_m = \left.\frac{1}{y^2}\right|_{y^1=-1/2, y^2=1/2} = 2 \quad , \quad \left.\frac{\partial x^2}{\partial y^2}\right|_m = -\left.\frac{y^1}{(y^2)^2}\right|_{y^1=y^2=1/2} = 2$$

and find that

$$\left.\frac{\partial}{\partial y^1}\right|_m = 2\left.\frac{\partial}{\partial x^2}\right|_m \quad , \quad \left.\frac{\partial}{\partial y^2}\right|_m = -4\left.\frac{\partial}{\partial x^1}\right|_m + 2\left.\frac{\partial}{\partial x^2}\right|_m$$

The dual space $(T_m M)^*$ will be important in our work and it will be called the **cotangent space** at m and denoted by $T^*_m M$. An element of $T^*_m M$ will be called a **tangent covector** at m (henceforth briefly **covector** at m.) Just as in the case of vectors, given a covector ω at m, we shall occasionally stress that it is a covector at the point m by writing it in the form ω_m or even $\omega|_m$.

Each differentiable f defined near m gives rise to a covector $df|_m$ at m defined to be the linear map that sends $\mathsf{u} \in T_m M$ to $\mathsf{u}_m f \in \mathbb{R}$. The covector $df|_m$ is called the **differential** of f at m.

In the presence of a chart $x = (x^i)_{1\le i\le n}$ with $m \in \mathrm{dom}(x)$, the differentials $dx^i|_m$ are of great significance.

THEOREM : For each $m \in M$, and each chart $x = (x^i)_{1\le i\le n}$ with $m \in \mathrm{dom}(x)$ the set

$$\mathfrak{B}^\star_{x,m} = \left\{ dx^i\big|_m \right\}_{1\le i\le n}$$

constitutes a basis for $T^\star_m M$. Indeed, $\mathfrak{B}^\star_{x,m}$ is the dual of the basis $\mathfrak{B}_{x,m}$ for $T_m M$ presented above. If f is a differentiable real valued function defined near m then

$$df|_m = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(m)\ dx^i|_m$$

Given charts x, y with $m \in \mathrm{dom}(x) \cap \mathrm{dom}(y)$, the elements of the bases $\mathfrak{B}^\star_{x,m}$ and $\mathfrak{B}^\star_{y,m}$ are related to one another by

$$dy^p|_m = \sum_{i=1}^{n} \frac{\partial y^p}{\partial x^i}(m)\ dx^i|_m$$

Proof : The first two assertions are obvious since

$$dx^i|_m(\left.\frac{\partial}{\partial x^j}\right|_m) = \frac{\partial x^i}{\partial x^j}(m) = \delta^i_j$$

The third assertion follows from

$$df|_m(\left.\frac{\partial}{\partial x^i}\right|_m) = \frac{\partial f}{\partial x^i}(m) \quad .$$

Finally, the last assertion is obtained by taking f to be y^p. ∎

PROBLEMS

I. Consider $\mathbb{S}^2 = \{(u, v, w) \in \mathbb{R}^3 \mid u^2 + v^2 + w^2 = 1\} \subseteq \mathbb{R}^3$ and the functions $f, g : \mathbb{S}^2 \longrightarrow \mathbb{R}$ defined by

$$f((u, v, w)) = w \ , \ g((u, v, w)) = v + u \ .$$

Find the unique vector $\mathsf{u} \in T_{(1,0,0)}\mathbf{S}^2$ such that $\mathsf{u}f = 1, \ \mathsf{u}g = 2$.

II. Consider $f : \mathbb{R}P^2 \longrightarrow \mathbb{R}$ defined by

$$f([u, v, w]) = \frac{uv}{u^2 + v^2 + w^2} \ .$$

Prove that there are exactly five points in $\mathbb{R}P^2$ at which the differential of f vanishes.

III. Remember that a real valued continuous function attains its extrema on a compact topological space.

(A) Let M be a manifold, $f : M \longrightarrow \mathbb{R}$ be a differentiable function. If $m \in M$ has a neighbourhood V such that $f(m) \leq f(m')$ for all $m' \in V$, prove that $df|_m = 0$.

(B)Let M be a compact manifold. If $g : M \longrightarrow \mathbb{R}$ is a nonconstant differentiable map, prove that the cardinality of the set

$$\{m \in M \mid dg|_m = 0 \in T^\star M\}$$

is at least two.