

HOMEWORK 7

1. Let $\mathbf{a} \times \mathbf{b}$ and $\langle \mathbf{a}, \mathbf{b} \rangle$ denote the vector product and inner product of $\mathbf{a}, \mathbf{b} \in \mathbb{R}_{vs}^3$, respectively. Consider the sphere

$$\mathbb{S}^2 = \{(u, v, w) \in \mathbb{R}^3 \mid u^2 + v^2 + w^2 = 1\} .$$

Let \mathbf{n} be the outwards directed unit normal field on \mathbb{S}^2 , to be precise

$$\mathbf{n}|_{(u,v,w)} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

for each $(u, v, w) \in \mathbb{S}^2$. Thus \mathbb{S}^2 is understood to have its usual structure as oriented manifold in the sense that for each $q \in \mathbb{S}^2$ the tangent vectors $\mathbf{a}, \mathbf{b} \in T_q\mathbb{S}^2$ constitute a right handed system iff $\det(\mathbf{a}, \mathbf{b}, \mathbf{n}_q) > 0$. With the usual identification

$$T_q\mathbb{S}^2 = \{\mathbf{x} \in \mathbb{R}^3 \mid \mathbf{x} \perp q\} \subseteq \mathbb{R}_{vs}^3$$

for each $q \in \mathbb{S}^2$, consider $\sigma \in \wedge^1(\mathbb{S}^2)$ defined by

$$\sigma(\mathbf{a}) = \left\langle \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{a} \times \mathbf{n} \right\rangle$$

for each $q \in \mathbb{S}^2$ and $\mathbf{a} \in T_q\mathbb{S}^2$.

(A) Prove that $\sigma = i_*(\Sigma)$ where $i : \mathbb{S}^2 \rightarrow \mathbb{R}^3$ is the identity imbedding and

$$\Sigma = (v - w)du + (w - u)dv + (u - v)dw .$$

(B) Compute

$$\int_{\mathbb{S}^2} df \wedge \sigma$$

where $f : \mathbb{S}^2 \rightarrow \mathbb{R}$ is defined by $f((u, v, w)) = v$ for each $(u, v, w) \in \mathbb{S}^2$.