HOMEWORK 6

1. Consider manifolds M, N and $h : M \longrightarrow N$ which is submersive, in the sense that

$$T_m h : T_m M \longrightarrow T_{f(m)N}$$

is surjective for every $m \in M$.

(A) If $\omega \in \Omega^p(N)$ vanishes at no point in $h(M) \subseteq N$, prove that $h_\star(\omega) \in \Omega^p(M)$ is nowhere vanishing.

(B) If M is not orientable, show that every $\lambda \in \Omega^r(N)$ vanishes at some point in $h(M) \subseteq N$, where $r = \dim N$.

2. The purpose of this problem is to present a diffeomorphism between $\mathbb{R}P^2$ and

$$SO(\mathbb{R}^3) = \{A \in \mathbb{R}^{3\times 3} \mid A^T A = I, \det(A) = 1\}$$

that arises naturally from certain properties of quaternions. Let $\mathbb{H}$ stand for the divison ring of quaternions. Remember that for any $\mathsf{q} = a + bi + cj + dk \in \mathbb{H}$

$$\mathrm{Re}(\mathsf{q}) = a \ .$$

$\mathsf{q} \in \mathbb{H}$ is said to be **purely imaginary** if $\mathrm{Re}(\mathsf{q}) = 0$. If $\mathsf{q} \neq 0$, then

$$\mathsf{q}^{-1} = \frac{a - bi - cj - dk}{a^2 + b^2 + c^2 + d^2} \quad .$$

(A) Remember that real numbers commute with quaternions. Prove that conjugation by quaternions send purely imaginary quaternions into purely imaginary quaternions. More precisely, prove that

$$\mathsf{q} i \mathsf{q}^{-1} = (a^2 + b^2 - c^2 - d^2)i + 2(bc + da)j + 2(bd - ca)k$$

for $\mathsf{q} = a + bi + cj + dk$. Similar expresions can be obtained for $\mathsf{q} j \mathsf{q}^{-1}$ and $\mathsf{q} k \mathsf{q}^{-1}$.

(B) Prove that $\mathrm{Re}(\mathsf{q}\mathsf{x}\mathsf{q}^{-1}) = \mathrm{Re}(\mathsf{x})$ for any $\mathsf{q}, \mathsf{x} \in \mathbb{H}$.

(C) Let $\langle \bullet, \bullet \rangle$ be the inner product on $\mathbb{H}$ as vector space over $\mathbb{R}$, which arises from the natural identification of $\mathbb{H}$ and $\mathbb{R}^4$ as vector spaces over $\mathbb{R}$. Prove that

$$\langle \mathsf{x}, \mathsf{y} \rangle = \mathrm{Re}(\mathsf{x}\overline{\mathsf{y}})$$

for any $\mathsf{x}, \mathsf{y} \in \mathbb{H}$.

(D) Prove that conjugation by quaternions is an isometry in $\mathbb{H}$. More precisely, show that

$$\langle \mathsf{q}\mathsf{x}\mathsf{q}^{-1}\,,\mathsf{q}\mathsf{y}\mathsf{q}^{-1}\rangle = \langle \mathsf{x}\,,\mathsf{y}\rangle$$

for any $\mathsf{q},\mathsf{x},\mathsf{y} \in \mathbb{H}$.

(E) Now it is a relatively simple task to check that the map $F : \mathbb{R}P^3 \longrightarrow SO(3)$ defined by

$$F([a,b,c,d]) = (a^2{+}b^2{+}c^2{+}d^2)^{-1} \begin{bmatrix} a^2+b^2-c^2-d^2 & 2(cb-da) & 2(db+ca) \\ 2(bc+da) & a^2-b^2+c^2-d^2 & 2(dc-ba) \\ 2(bd-ca) & 2(cd+ba) & a^2-b^2-c^2+d^2 \end{bmatrix}$$

is a diffeomorphism.