Math 505

Differentiable Manifolds

FINAL EXAMINATION

1. Consider $f : N \longrightarrow P$ and let $p \in P$ be a regular value of f. Of course, $M = f^{-1}(p)$ is a submanifold of N.

(A) Prove that for each $m \in M$, the tangent space $T_m M$ can be identified with $\ker(T_m f) \subseteq T_m N$. Quite precisely, show that

$$T_m i(T_m M) = \ker(T_m f) \subseteq T_m N \ .$$

where $i : M \longrightarrow N$ is the inclusion map that imbeds M into N .

(*Hint* : Remember that $\mathsf{u} \in T_m N$ lies in $T_m M$ iff there exists $\lambda : J \subseteq_{\mathrm{op}} \mathbb{R}$ with $0 \in J$ such that $\lambda(0) = m$, $f(\lambda(t)) = p$ for all $t \in J$ and

$$\mathsf{u} = \dot{\lambda}(0) = T_0 \lambda\Big(\frac{\partial}{\partial t}\Big|_{t=0} \Big) \ . \ )$$

(B) Prove that

$$M = \{(x, y, z, w) \in \mathbb{R}^4 \mid x^2 + y^2 = z^2 + w^2 = 1\}$$

is a compact submanifold of $\mathbb{R}^4$.

(C) Prove that M is diffeomorphic to the two dimensional torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$.

(D) With the standard identifications mentioned above, prove that the vectors

$$\mathsf{a} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \ , \ \mathsf{b} = -w\frac{\partial}{\partial z} + z\frac{\partial}{\partial w} \in T_{(x,y,z,w)}\mathbb{R}^4$$

constitute a basis for $T_{(x,y,z,w)}M$ for any $(x, y, z, w) \in M$

(E) Prove that the covectors

$$\alpha = -ydx + xdy \ , \ \beta = -wdz + zdw \in T^{\star}_{(x,y,z,w)}\mathbb{R}^4$$

constitute the dual of the basis $\{\mathsf{a}, \mathsf{b}\}$ of $T_{(x,y,z,w)}M$.

(F) Prove that M is orientable. (*Hint* : Consider $\alpha \wedge \beta \in \Omega^2(M)$.)

(G) Compute

$$\int_M \alpha \wedge \beta \ .$$

(*Hint* : You might like to use the chart (θ, φ) on M with $x = \cos\theta$, $y = \sin\theta$, $z = \cos\varphi$, $w = \sin\varphi$. Describe its domain carefully.)

(H) Is $\alpha \wedge \beta \in \Omega^2(M)$ exact ? (*Hint*: Notice that the integral of an exact form on a manifold without boundary must vanish.)

2. Consider the 3−dimensional sphere

$$\mathbb{S}^3 = \{(x, y, z, w) \in \mathbb{R}^4 \mid x^2 + y^2 + z^2 + w^2 = 1\} \ .$$

For each $(x, y, z, w) \in \mathbb{S}^3$, as usual $T_{(x,y,z,w)}\mathbb{S}^3$ is identified with the elements of $\mathbb{R}^4_{\text{vs}}$ which are perpendicular to

$$\mathsf{n} = \mathsf{n}_{(x,y,z,w)} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

(A) Let g be the Riemannian tensor field that $\mathbb{S}^3$ inherits from the ambient space $\mathbb{R}^4$ with its standard structure as inner product space. Prove that for any $(x, y, z, w) \in \mathbb{S}^3$,

$$\mathrm{Vol}_{\mathsf{g}}\big|_{(x,y,z,w)}(\mathsf{a}, \mathsf{b}, \mathsf{c}) = \det\Big(\mathsf{n}_{(x,y,z,w)}, \mathsf{a}, \mathsf{b}, \mathsf{c}\Big)$$

for any $\mathsf{a}, \mathsf{b}, \mathsf{c} \in T_{(x,y,z,w)}\mathbb{S}^3$.

(B) Compute the volume of $\mathbb{S}^3$ with respect to its standard structure as oriented Riemannian manifold. (*Hint*: You might like (or not) to use the chart (ξ, φ, θ) on M with $x = \sin\xi \sin\varphi \cos\theta$, $y = \sin\xi \sin\varphi \sin\theta$, $z = \sin\xi \cos\varphi$, $w = \cos\xi$. Describe its domain carefully.)

3. We have defined the exterior derivative by means of local formulae and shown how these can be glued together to give rise to a global concept. For any manifold M, prove the following global formulae :

(A)

$$d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega(\,[X, Y]\,)$$

for any $\omega \in \Omega^1(M)$ and any $X, Y \in \mathfrak{X}(M)$. (*Hint* : It will be sufficient to check that in the presence of a chart $x = (x^i)_{1 \le i \le n}$ the substitutions

$$X = \frac{\partial}{\partial x^i}\ , Y = \frac{\partial}{\partial x^j}$$

and

$$\omega\big|_{\mathrm{dom}(x)} = \sum_{1 \le < i < j \le n} \omega_{ij}\, dx^i \wedge dx^j$$

give rise to the local definitions.)

(B)

$$d\lambda(X,Y,Z) = X\lambda(Y,Z) + Y\lambda(Z,X) + Z\lambda(X,Y) - \lambda(\,[X,Y],Z) - \lambda(\,[Y,Z],X) - \lambda(\,[Z,X],Y)\ .$$

for any $\lambda \in \Omega^2(M)$ and any $X,Y,Z \in \mathfrak{X}(M)$.

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You might also like to take this opportunity to look up the general formulae in any standard text book and decide for yourself whether I have been pedagogically amiss in starting with the local definition. Of course the global definition is very glamorous and direct and clear. Yet all these desirable qualities obtain at the cost of transparency.

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4. Consider $\mathbb{R}^{n\times n}$ with its usual structur as vector space.

(A) Let $\mathfrak{S} = \{A \in \mathbb{R}^{n\times n} \mid A^T = A\}$. Prove that $\mathfrak{S}$ is a subspace of $\mathbb{R}^{n\times n}$ and

$$\dim_{\mathbb{R}}(\mathfrak{S}) = \frac{n(n+1)}{2} \qquad .$$

(B) Let $\mathfrak{H} : \mathbb{R}^{n\times n} \longrightarrow \mathfrak{S}$ be defined by

$$\mathfrak{H}(X) = X^T X\ .$$

Prove that for any $P \in \mathbb{R}^{n\times n}$ the directional derivative $D_P\mathfrak{H}$ of $\mathfrak{H}$ at P is given by

$$D_P F(A) = P^T A + A^T P$$

for each $A \in \mathbb{R}^{n\times n}$.

(C) Prove that for each $S \in \mathfrak{S}$,

$$X = \frac{1}{2}\,(P^T)^{-1} S$$

is a solution of

$$P^T X + X^T P = S\ .$$

(D) Prove that $I \in \mathfrak{S}$ is a regular value of $\mathfrak{S}$.

(E) Prove that

$$O(\mathbb{R}^n) = \{P \in \mathbb{R}^{n\times n} \mid P^T P = I\}$$

is a compact submanifold of $\mathbb{R}^{n\times n}$. (*Hint* : Notice that $O(\mathbb{R}^n)$ is a bounded set with respect to the natural inner product on $\mathbb{R}^{n\times n}$.)

(F) Since $O(\mathbb{R}^n)$ is a submanifold of $\mathbb{R}^{n\times n}$ we may identify each $T_PO(\mathbb{R}^n)$ with a subspace of $\mathbb{R}^{n\times n}_{vs}$. Prove that $A \in \mathbb{R}^{n\times n}_{vs}$ is an element of $T_PO(\mathbb{R}^n)$ iff $PA^TP = -A$. (*Hint* : Take a curve $\lambda(t) \in O(\mathbb{R}^n)$ with $\lambda(0) = P$, $\dot{\lambda}(0) = A$. Differentiate the equality $\lambda^T\lambda = I$.)

(G) Let $\mathrm{L}_P : O(\mathbb{R}^n) \longrightarrow O(\mathbb{R}^n)$ and $\mathrm{R}_P : O(\mathbb{R}^n) \longrightarrow O(\mathbb{R}^n)$ be the left and right translations by $P \in O(\mathbb{R}^n)$ defined by

$$\mathrm{L}_P(Q) = PQ \quad , \quad \mathrm{R}_P(Q) = QP$$

for each $Q \in O(\mathbb{R}^n)$. A vector field $X \in \mathfrak{X}(O(\mathbb{R}^n))$ is said to be *left invariant* if $T\mathrm{L}_P(X_Q) = X_{PQ}$. One can define *right invariant* elements of $\mathfrak{X}(O(\mathbb{R}^n))$ similarly. Prove that each left invariant $X \in \mathfrak{X}(O(\mathbb{R}^n))$ is of the form $\mathbb{V}_A$ for some $A \in \mathbb{R}^{n\times n}$ with $A^T = -A$ where

$$\mathbb{V}_A|_Q = QA$$

for $Q \in O(\mathbb{R}^n)$.

(H) Prove that

$$[\mathbb{V}_A, \mathbb{V}_B] = \mathbb{V}_{AB-BA}$$

(I) If $X \in \mathfrak{X}(O(\mathbb{R}^n))$ is left invariant and $Y \in \mathfrak{X}(O(\mathbb{R}^n))$ is right invariant, prove that $[X, Y] = 0$.

5. Let U be a finite dimensional vector space space over $\mathbb{R}$. Consider linearly independent vectors $\mathsf{e}_1, \mathsf{e}_2, \cdots, \mathsf{e}_r \in \mathrm{U}$.

(A) For any $\mathsf{v}_1, \mathsf{v}_2, \cdots, \mathsf{v}_r \in \mathrm{U}$ prove that the following are equivalent :

(i) $\displaystyle\sum_{k=1}^{r} \mathsf{e}_k \wedge \mathsf{v}_k = 0$.

(ii) There exist $a_{ij} \in \mathbb{R}$ with $a_{ij} = a_{ji}$ such that

$$\mathsf{v}_k = \sum_{l=1}^{r} a_{kl}\mathsf{e}_l$$

for each $1 \le l \le r$. (*Hint* : Extend $\mathsf{e}_1, \mathsf{e}_2, \cdots, \mathsf{e}_r \in \mathrm{U}$ to a basis $\{\mathsf{e}_i\}_{i=1}^n$ of U and put

$$\mathsf{v}_k = \sum_{l=1}^{n} a_{kl}\mathsf{e}_l$$

(B) Given $W \in \bigwedge^q(\mathrm{U})$, show that there exist $A_1, A_2, \cdots A_r \in \bigwedge^{q-1}(\mathrm{U})$ with

$$W = \sum_{k=1}^{r} \mathsf{e}_k \wedge A_k$$

iff

$$\mathsf{e}_1 \wedge \mathsf{e}_2 \wedge \cdots \mathsf{e}_r \wedge W = 0 \ .$$