SELECTED PROBLEMS VIII

1. What is the image of

(a) the line $y = 0$ under $\mathrm{Inv}_{(2,3),6}$,

(b) the circle $x^2 + y^2 = 1$ under $\mathrm{Inv}_{(1,1),6}$?

2. Consider an inversion φ of centre $(1, 2)$ with power 3. What is the image of

(a) the point $(-1, 3)$

(b) the line $x + y = 3$

(c) the line $2x - 3y = 1$

(d) the circle $x^2 + y^2 = 1$

(e) the circle $x^2 + y^2 = 5$

under φ?

3. (a) What is the image of the line the line $x = -1$ under an inversion of centre $(8, 1)$ and power 36 ?

(b) What is the image of the circle $x^2 + y^2 - 4x = 0$ under an inversion of centre $(2, -2)$ and power 1 ?

4. Prove that the *"limaçon"*

$$r = a + b\cos\theta \qquad (a, b > 0)$$

is mapped into a conic section under the inversion in the circle $r = a$. Compute the eccentricity of the conic. What happens in the special case $a = b$ (the *"cardioid"*)?

5. Prove that if P is a point outside a circle α, then the inversion with respect to α

maps P to a point on the chord joining the points of contact of the tangents from P to α.

6. Let α, β be orthogonal circles intersecting in distinct points A, B. Let $C \in \alpha - \{A, B\}$, $D \in \beta - \{A, B\}$. Prove that the circumcircles of the triangles ADC, CBD are orthogonal circles.

7. Let α, β be circles tangent to each other at O. Let the tangent to α at $A \in \alpha$ intersect β in points B, C. Prove that OA is a bisector of the angle $< BOC$.

(Hint : Take an inversion of centre O.)

8. Let A be a point on a circle α with centre C. The image of α under an inversion with centre A intersects AC at B. Prove that if D is the inverse of C then $|AB| = |BD|$.

9. Consider circles α, β intersecting in points P, Q. Let ω be a circle which is perpendicular to both α and β. Prove that any circle through P, Q is perpendicular to ω.

10. Let α, β, γ be circles through a point O with $\beta \cap \gamma = \{O, A\}$, $\gamma \cap \alpha = \{O, B\}$, $\alpha \cap \beta = \{O, C\}$. Let ω be a circle tangent to α, β, γ. Let α', β', γ' be the circles through O, inciding respectively in A, B, C and all orthogonal to ω. Prove that α', β', γ' have a common point other than O.

11. (a) Consider circles γ_1, γ_2 of centres O_1, O_2 and radii r_1, r_2 respectively. If P is a point with

$$d^2 = |O_1P|^2 - r_1^2 = |O_2P|^2 - r_2^2$$

prove that the circle of centre P and radius d is orthogonal to both γ_1 and γ_2.

(b) Prove that for any non - intersecting circles γ_1, γ_2, there exist at least two intersecting (generalised) circles orthogonal to both γ_1 and γ_2.

(c) Prove that any two non - intersecting circles can be mapped into concentric circles by an inversion.

12. Let A, B, C, D be distinct concyclic points, C and D separated by A and B.

If p_1, p_2, p_3 are the lengths of the perpendiculars from D to the lines AB, BC, CA respectively, prove that

$$\frac{|AB|}{p_1} = \frac{|BC|}{p_2} + \frac{|CA|}{p_3}.$$

13. Let Ω be a circle of diameter $[BC]$. Consider $A \in \Omega$ such that ABC is an isosceles triangle. For any $X \in \Omega$ not on the same side of BC as A, (using the theorem of Ptolemy or otherwise) prove that

$$|XA| = \frac{1}{\sqrt{2}}(|XB| + |XC|)$$

Obtain the corresponding formula for the situation in which X is on the same side of BC as A.

14. Using the theorem of Ptolemaus or otherwise compute the lengths of the diagonals of a regular pentagon of unit sidelength.

15. Let A, B, C, D and $O \neq A, B, C, D$ be points on a line k, A', B', C', D' be the respective images of A, B, C, D with respect to an inversion of centre O. Prove that

$$\frac{CA}{CB} \div \frac{DA}{DB} = \frac{C'A'}{C'B'} \div \frac{D'A'}{D'B'}.$$