# SELECTED PROBLEMS VII

**1.** (a) What is the image of the line $2x + y = 2$ under $T_{[-1,3]}$ ?

(b) What is the image of the line $x + y = 1$ under $Ref_k$ , where $k$ is the line $y = 2x$ ?

(c) What is the image of the line $x = 1$ under $Rot((0, 0), \pi/4)$ ?

**2.** (a) Let $P$, $Q$ be distinct points. Prove that half-turns in $P$ and $Q$ together give rise to a translation. Which ?

(b) Consider a convex positively oriented pentagon $ABCDE$ with positively oriented squares $BAPQ$, $CBRS$, $DCTU$, $EDVW$, $AEXY$ constructed on its sides with respective centers $M_a$, $M_b$, $M_c$, $M_d$, $M_e$. What is the image of $A$ under the isometry

$$Rot(M_d, -\frac{\pi}{2}) \circ Rot(M_c, -\frac{\pi}{2}) \circ Rot(M_b, -\frac{\pi}{2}) \circ Rot(M_a, -\frac{\pi}{2}) \quad ?$$

(c) What is the image of $Y$ under the same isometry ?

**3.** Consider a convex positively oriented quadrangle $ABCD$ with positively oriented equilateral triangles $BAP$, $CBQ$, $DCR$, $ADS$, constructed on its sides (i. e. on the "outside"). Let

$$\alpha = Rot(P, \frac{\pi}{3}) \circ Rot(Q, \frac{\pi}{3}) \circ Rot(R, \frac{\pi}{3})$$
$$\beta = Rot(Q, \frac{\pi}{3}) \circ Rot(R, \frac{\pi}{3}) \circ Rot(S, \frac{\pi}{3})$$

(a) Prove that $\alpha^2 = Id$, $\beta^2 = Id$.

(b) Quite generally, what is the composition of two half-turns ?

(c) Let $A'$ be the image of $A$ under $\beta \circ \alpha$. What can you say about the quadrangle $AA'BD$ ?

**4.** Consider a triangle $ABC$ and squares $BPQC$, $CRSA$, $AUVB$ which have the same orientation as $ABC$. Let $M_a$ $M_b$ $M_c$ be the respective centres of the squares $BPQC$, $CRSA$, $AUVB$. Prove that $AM_a$, $BM_b$, $CM_c$ are the altitudes of the triangle $M_aM_bM_c$.

**5.** Let $ABCD$ be a quadrangle, $ABPQ$, $BCRS$, $CDTU$, $DAVW$ be squares with the same orientation as $ABCD$. having respective centres $M_1$, $M_2$, $M_3$, $M_4$. Prove that the lines $M_1M_3$, $M_2M_4$ are perpendicular to each other.

**6.** Given a triangle $ABC$, prove that it is possible to construct (with ruler and compasses) points $X \in AB$, $Y \in AC$ such that $|BX| = |XY| = |YC|$.

**7.** Given distinct points $A$, $B$, $C$, $D$ on a circle $\gamma$ prove that it is possible to construct (with ruler and compasses) a point $X \in \gamma$ such that $AX$, $BX$ separates a line segment of given length on $CD$.

**8.** Given distinct points $A$, $B$, $C$, $D$ on a circle $\gamma$ and $J \in [C, D]$ prove that it is possible to to construct (with ruler and compasses) a point $X \in \gamma$ such that $AX$, $BX$ separates on $CD$ a line segment whereof the midpoint is $J$.

**9.** Given a line $k$ and points $A$, $B$ lying on the same side of $k$ prove that it is possible to construct (with ruler and compasses) a point $X$ on $k$ such that $< (AX, k) =< (k, BX)$.

**10.** Given lines $k, l$ and points $A$, $B$ prove that it is possible to construct (with ruler and compasses) a point $X$ on $k$ such that the $< (l, AX) =< (BX, l)$.

**11.** Given a line $k$ and points $A$, $B$ lying on the same side of $k$ prove that it is possible to construct (with ruler and compasses) a point $X$ on $k$ such that $|AX|+|BX|$ equals a given length.

The same problem with $|AX| + |BX|$ replaced by $|AX| - |BX|$.

**12.** (a) What is the image of the circle $x^2 + y^2 - 2y - 3 = 0$ under $Hom((2,1),3)$ ?

(b) What is the image of the circle $x^2 + y^2 - 2x - 8 = 0$ under the half turn of center $(5,1)$ ?

(c) What is the image of the line $x + 2y - 5 = 0$ under $Hom((1,1),3)$ ?

**13.** Given distinct points $X$, $Y$, $Z$ let

$$Hom(X, \lambda)(B) = C$$

$$Hom(Y, \mu)(C) = A$$

$$Hom(Z, \nu)(A) = B$$

where $\lambda, \mu, \nu \notin \{0,1\}$ write down and prove a necessary and sufficient condition in terms of $\lambda$, $\mu$, $\nu$ for $X$, $Y$, $Z$ to be collinear.

State the analogous condition for $AX$, $BY$, $CZ$ to be concurrent.

**14.** Let $ABC$ be a triangle with orthocenter $H$, centroid $G$, and circumcircle $O$.

(a) What is the image of $ABC$ under $Hom(G, -1/2)$ ?

(b) Prove that $Hom(G, -1/2)(H) = O$ .

(c) Let $O_a$, $O_b$, $O_c$ be the circumcenters of the triangles $HBC$, $HCA$, $HAB$. Prove that $O_a$, $O_b$, $O_c$ are the reflections of $O$ in $BC$, $CA$, $AB$ respectively.

(d) Prove that $|OO_a| = |AH|$, $|OO_b| = |BH|$, $|OO_c| = |CH|$.

(e) Prove that a half-turn sends $ABCH$ into $OO_aO_bO_c$. Which ?