SELECTED PROBLEMS VI

1. (a) Consider a line d and a point $F \notin d$. Let circles ξ, η through F touch d at X, Y respectively and $\xi \cap \eta = \{F, M\}$. Prove that FM bisects $[XY]$.

(b) Prove that in a parabola the reflection of the focus in a tangent lies on the directrix.

(c) Consider a parabola φ, points $A, B \in \varphi$. Prove that the tangents to φ at A, B intersect on the directrix of φ iff AB goes through the focus of φ.

2. Let φ be an ellips with foci F, F'. For each $X \in \varphi$, let γ_x be the circle of diameter $[XF]$.

a) Prove that γ_x is tangent to the circle whereof the centre coincides with the centre of φ and the radius equals the major radius of φ of φ.

b) Let the tangent and normal to φ at X intersect γ_x in T, N respectively. Prove that TN goes through the centre of φ. (Equivalently, it bisects $[FF']$.)

3. Let φ be an ellipse with major axis k (the line joining the foci), minor axis k' (the perpendicular bisector of the line segment joining the foci). For any $X \in \varphi$ let t and n be the tangent and the normal to φ at X. If $t \cap k = \{T\}$, $t \cap k' = \{T'\}$, $n \cap k = \{N\}$, $n \cap k' = \{N'\}$ prove that

a) the circumcircle of $XT'N'$ goes through the foci of φ,

b) the circumcircles of XTN and $XT'N'$ are perpendicular to each other.

4. Consider an ellipse φ with foci F, F'. Let a line through F meet φ in X, X'. If the normals to φ at X, X' intersect in N, prove that the parallel to FF' through N bisects $[XX']$.

5. Let φ be an ellipse with foci F, F'. Prove that for any $X, X' \in \varphi$ the lines $XF, XF', X'F, X'F'$ are tangent to the same circle.

6. Let φ be an ellipse with foci F, F'. Let t, t' be the tangents to φ through a point P "outside" φ. If H, H' are respectively the feet of the perpendiculars from F on t, t', prove that PF' is perpendicular to HH'.

7. Let φ be an ellipse of foci F, F'. For any $X \in \varphi$, let XF, XF' intersect φ in P, Q for a second time, respectively. Let the tangents to φ at P, Q intersect in N. Prove that XN is normal to φ at X.

8. Let φ be an ellipse with a directrix circle Γ. Prove that, when the tangents to φ, through $A \in \Gamma$ meet Γ in B, C for a second time, BC is tangent to φ. Show also, that as A changes the orthocentre and the nine-point circle of ABC remains invariant.

9. Let φ, ψ be ellipses with common foci. Prove that the locus of the point of intersection of tangents to φ, ψ which are perpendicular to each other, is a circle.

10. Let φ be an ellipse of foci F, F'. Let $\varphi \cap FF' = \{A, A'\}$. Let ℓ, ℓ' be the tangents to φ. at A, A'. For any tangent t to φ let $t \cap \ell = \{P\}, t \cap \ell' = \{P'\}$. Prove that

a) $< (FP, FP') =< (F'P, F'P') = \pi/2$

b) $FP, F'P'$ intersect on the normal to φ at the point of tangency of t.

11. Let φ be an ellipse of foci, F, F'. Consider $X \in \varphi$. Let Q, Q' be the points in which the normal of φ at X intersects the perpendiculars to XF, XF' erected at F, F' respectively. Prove that, the perpendicular bisector of $[FF']$ (the minor axis of φ) bisects $[Q, Q']$.

12. Consider a variable point X on an ellipse φ of foci F, F'. Let t_x be the tangent to φ at X. Let $A_X \in t_X$ be a point with the property that the second tangent to φ through A_X touches the ellipse at a point X' with $FX \parallel F'X'$. What is the locus of A_X?