SELECTED PROBLEMS V

1. (a)What is the locus of a point whereof the power with respect to a fixed circle is a constant ?

(b) Consider fixed non-concentric circles Γ_1, Γ_2. Prove that the locus of a point whereof the powers with respect to Γ_1 and Γ_2 differ by a constant is a line parallel to the radical axis of Γ_1 and Γ_2.

(c) Prove that the locus of a point whereof the ratio of powers with respect to two given circles Γ_1, Γ_2 is a constant i s a circle the centre of which lies on the line joining centres of Γ_1, Γ_2.

2. (a) Given two orthogonal circles, is it possible for the centre of one to lie on the other ?

(b) Let each one of the circles C_1, C_2 intersect each one of the circles Γ_1, Γ_2 orthogonally. Prove that the radical axis of C_1, C_2 is the line joining the centres of Γ_1, Γ_2.

3. Prove that the radical axis of two circles which have a tangent line in common bisects the line segment of the tangent line determined by the points of tangency.

4. Given a circle γ and coaxal circles Γ_1, Γ_2, Γ_3, prove that the radical axes of γ and Γ_1, γ and Γ_2, γ and Γ_3 are concurrent.

5. Given triangle ABC, let $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ be the feet of the perpendiculars from A, B, C on BC, CA, AB respectively. Let BC and $\tilde{B}\tilde{C}$, CA and $\tilde{C}\tilde{A}$, AB and $\tilde{A}\tilde{B}$ meet in X, Y, Z respectively.

(a) Prove that X, Y, Z are collinear by employing the theorems of Menelaus and Ceva.

(b) Prove the same result by demonstrating that X, Y, Z lie on the radical axis of the circumcircle and the 9-point-circle of ABC. (*Hint* : Consider the circle with diameter $[BC]$.)

(c) Prove that the line containing X, Y, Z is perpendicular to the Euler line of ABC.

6. Introducing the *Spieker point* of a triangle ABC:

Let (I) touch BC, CA, AB in S, T, U respectively. Let (I_a) and (I_b) and (I_c) touch BC, CA, AB in S_a, T_a, U_a and S_b, T_b, U_b and S_c, T_c, U_c respectively. Let Y, Z be the respective midpoints of the line segments $[T_b, T_c]$, $[U_b, U_c]$.

(a) Suppose that (I_b) and (I_c) are not congruent. (Y and Z are distinct !) Employ the thorem of Menelaus to prove that YZ bisects the line segment $[B, C]$.

(b) Compute the powers of Y, Z with respect to the circles (I_b) and (I_c).

(c) Suppose that (I_b) and (I_c) are not congruent. Prove without using the thorem of Menelaus that YZ bisects the line segments $[B, C]$ and $[S_b, S_c]$.

(d) Let A', B', C', be the midpoints of $[B, C]$, $[C, A]$ $[A, B]$. Prove that the incenter of $A'B'C'$ is the radical center of (I_a), (I_b), (I_c).

$\diamond$ The incenter of $A'B'C'$ is called the *Spieker point* of ABC. We shall denote it by Σ. $\diamond$

(e) Prove that Σ is the center of gravity of the set $[B, C] \cup [C, A] \cup [A, B]$.

(*Hint :* Concentrate each edge of the triangle into a point lying at its midpoint. Let that point have a mass proportional to the length of the edge.)

7. (a) In the triangle ABC, let AH, BH, CH meet BC, CA, AB in $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ respectively and observe that

$$HA \cdot H\tilde{A} = HB \cdot H\tilde{B} = HC \cdot H\tilde{C}.$$

(b) Let A', B', C' be midpoints of $[B, C]$, $[C, A]$, $[A, B]$ respectively. Consider $X \in BC - \{B, C\}$, $Y \in CA - \{C, A\}$, $Z \in AB - \{A, B\}$ such that AX, BY, CZ are concurrent. If X', Y', Z' are midpoints of $[A, X]$, $[B, Y]$, $[C, Z]$ respectively, prove that $X'A'$, $Y'B'$, $Z'C'$ concur in a point Υ.

(c) Let α, β, γ be the respective circles of diameter $[B, C]$, $[C, A]$, $[A, B]$. Let α', β', γ' be the respective circles of diameter $[A, X]$, $[B, Y]$, $[C, Z]$. Let $\alpha \cap \alpha' = \{S, S'\}$, $\beta \cap \beta' = \{T, T'\}$, $\gamma \cap \gamma' = \{U, U'\}$. Prove that the six points S, S', T, T', U, U' lie on a circle of center Υ.

8. (a) Given a triangle UVW and points $P \in UW$, $Q \in UV$, let β, γ be circles with respective diameters $[V, P], [W, Q]$. Prove that the orthocenter of UVW lies on the radical axis of β and γ.

(b) Consider a quadrangle $ABCD$ with $AC \cap BD = \{F\}$. Let K, L be the orthocenters of AFD, BFC, let S, T be the midpoints of $[A, B], [C, D]$ respectively. Prove that $KL \perp ST$. (*Hint :* Circles of diameters $[A, B]$,$[C, D]$.)

(c) Let Y, Z be the respective centroids of CFD, AFB. prove that $KL \perp YZ$.