SELECTED PROBLEMS IV

1. In a triangle ABC with orthocenter H and circumcenter O , let A' denote the midpoint of $[B, C]$.

(a) Prove that $|AH| = 2|A'O|$.

(b) Prove that the circumcenter of HBC is the reflection of O in BC .

(c) Consider distinct points K, L, M, N on a circle Γ . Let T, U, V, W be the respective orthocenters of the triangles LMN, KMN, KLN, KLM. Prove that the line segments $[K, T]$, $[L, U]$, $[M, V]$, $[N, W]$ have a common midpoint .

(d) Prove that the respective Simson lines of K, L, M, N, with respect to LMN, KMN, KLN, KLM are concurrent.

2. Let ABC be a triangle with orthocenter H. Prove that a Simson line of ABC is Simson line of HBC, HCA, HAB as well.

3. Given a triangle ABC and lines λ, μ, prove that the following statements are equivalent :

(i) λ, μ are Simson lines of ABC which are perpendicular to one another.

(ii) λ, μ are Simson lines of ABC which are isotomic conjugates with respect to ABC.

(iii) λ, μ are are isotomically conjugate with respect to ABC and perpendicular to one another.

4. In a triangle ABC, for each $X \in (O)$ let l_X denote the Simson line of X with respect to ABC.

(a) Prove that for any $X \in (O)$, l_X is parallel to AY where $[XY]$ is the unique chord of (O) perpendicular to BC.

(b) Given $X \in (O)$, prove that in order for l_X to be parallel to OX it is necessary and sufficient that

$$3 \overset{\frown}{XA} = B - C \ .$$

(c) Prove that there exist exactly three points X on (O) with the property $(\star)$. Show that these points constitute an equilateral triangle.

(d) Prove that the Simson lines in question concur in the Euler point of ABC . (*Hint* : Remember that l_X bisects $[HX]$.)

5. In a triangle ABC let $M \in (O)$, k be the Simson line of M with respect to ABC. Let $\alpha, \beta, \gamma, \delta \in k$ denote the respective feet of the the perpendiculars from M onto BC, CA, AB, k.

(a) Prove that

$$|MB||MC| = 2R|M\alpha| \ .$$

(b) Prove that

$$|M\beta||M\gamma| = 2\rho|MA|$$

where $|M\delta| = 2\rho$.

(c) Prove that

$$|MA||M\alpha| = |MB||M\beta| = |MC||M\gamma| = 4R\rho \ .$$

(d) Prove that

$$|MA||MB||MC| = 8R^2\rho \ .$$

(e) Prove that

$$|M\alpha||M\beta||M\gamma| = 8R\rho^2 \ .$$