SELECTED PROBLEMS

III

1. Given a triangle ABC, let A' be the midpoint of $[B,C]$ and consider $Y \in CA - \{C,A\}$, $Z \in AB - \{A,B\}$ such that BY and CZ meet on AA'. Prove that YZ is parallel to BC.

2. Given a triangle ABC, consider $P \in BC - \{B,C\}$, $Q \in CA - \{C,A\}$, $R \in AB - \{A,B\}$ such that AP, BQ, CR are concurrent. Let QR, RP, PQ meet BC, CA, AB in X, Y, Z respectively. Prove that

(a) X, Y, Z are collinear.

(b) AP, BY, CZ are concurrent.

3. Given a quadrilateral $ABCD$, consider $X \in AB - \{A,B\}$, $Y \in BC - \{B,C\}$, $Z \in CD - \{C,D\}$, $T \in DA - \{D,A\}$.

(a) Prove that if X, Y, Z, T are collinear then

$$\frac{XA}{XB} \cdot \frac{YB}{YC} \cdot \frac{ZC}{ZD} \cdot \frac{TD}{TA} = 1 \quad .$$

Is the converse true ?

(b) Given a triangle ABC, let A', B', C' be midpoints of $[B,C]$, $[C,A]$, $[A,B]$ respectively. Consider $L \in BC - \{B,C\}$, $M \in CA - \{C,A\}$, $N \in AB - \{A,B\}$ such that AL, BM, CN are concurrent. If X, Y, Z are midpoints of $[M,N]$ $[N,L]$, $[L,M]$ respectively prove that AX, BY, CZ are concurrent.

4. Given a triangle ABC, let A', B', C' be midpoints of $[B,C]$, $[C,A]$, $[A,B]$ respectively. Consider $L \in BC - \{B,C\}$, $M \in CA - \{C,A\}$, $N \in AB - \{A,B\}$ such that AL, BM, CN are concurrent. If P, Q, R are midpoints of AL BM, CN, respectively prove that PA', QB', RC' are concurrent.

5. If three sides of a variable triangle pass through three fixed collinear points while two vertices move along fixed lines, prove that the third vertex moves along a line concurrent with the other two. (*Hint* : Desargues' Theorem)

6. Consider a quadrangle $ABCD$ with points P, Q, R, S, T on BD, AB, AD, CB, CD respectively. Prove that if P, Q, R are collinear and P, S, T are collinear, then the lines QS, RT, AC are concurrent. (*Hint* : Desargues' Theorem)

7. Given a triangle ABC, let D be a point on AB. Consider points H, K not on any side of ABC. If P is a variable point on CD and AP intersects DH in Q, BP intersects DK in R, prove that QR meets AB in a fixed point (*Hint* : Desargues' Theorem)

8. The following is a problem by the late Prof. Demir who liked to present it with a quaintly arithmetic notation which is very suggestive and delightful :

Given distinct non-collinear points $1, 2, 3, 4$ consider points $12 \in 1 \cdot 2 - \{1, 2\}$, $23 \in 2 \cdot 3 - \{2, 3\}$, $34 \in 3 \cdot 4 - \{3, 4\}$. Let $1 \cdot 23$, $12 \cdot 3$ and $2 \cdot 34$, $23 \cdot 4$ meet in 123 and 234. Finally, let $1 \cdot 234$ intersect $123 \cdot 4$ in 1234. Prove that 12, 1234, 34 are collinear. (*Hint* : Desargues' Theorem)

9. Given triangle ABC, let D be a point on BC, P, Q points on AB. Let PD meet AC in H, QD meet AC in K, CP meet AD in M, CQ meet AD in N. Prove that KM and HN meet on AB. (*Hint* : The dual of the Pappus Theorem)

10. Consider a triangle ABC with points D, E, F on BC, CA, AB respectively. A variable line through A meets DE and DF in P and Q respectively. Prove that the lines BP and CQ meet on a fixed line. (*Hint* : The dual of the Pappus Theorem)

11. Let ABC, DEF be triangles with AD, BE, CF concurrent. Prove that if AE, BF, CD are concurrent so are AF, BD and CE. (*Hint* : The dual of the Pappus Theorem)

12. Introducing *isotomic conjugacy* with respect to a triangle ABC:

(a) Consider $X \in BC - \{B, C\}$, $Y \in CA - \{C, A\}$, $Z \in AB - \{A, B\}$. Let $X' \in BC - \{B, C\}$, $Y' \in CA - \{C, A\}$, $Z' \in AB - \{A, B\}$. be reflections of X, Y, Z in the respective midpoints of $[B, C]$ $[C, A]$, $[A, B]$. Prove that AX, BY, CZ are concurrent or parallel iff AX', BY', CZ' are concurrent or parallel.

$\diamond$ If AX, BY, CZ concur in M (an ordinary point or a point "at infinity") and

AX', BY', CZ' concur in M° (again an ordinary point or a point "at infinity"), the points M and M° are said to be *isotomic conjugates* of one another with respect to the triangle ABC. $\diamond$

(b) Prove that the areas of the triangles XYZ and $X'Y'Z'$ are equal.

(c) Prove that X', Y', Z' are collinear iff X, Y, Z are collinear.

$\diamond$ If X, Y, Z lie on the line m and X', Y', Z' lie on the line m° the lines m and m° are said to be *isotomic conjugates* of one another with respect to the triangle ABC.$\diamond$

13. Introducing the *Nagel and Gergonne points* of a triangle ABC:

(a) Let (I) touch BC, CA, AB in S, T, U respectively. Prove that AS, BT, CU concur.

$\diamond$ The point in which AS, BT, CU concur is a remarkable point of ABC, called the *Gergonne point* of ABC. It is usually denoted by Γ. $\diamond$

(b) Let (I_a) and (I_b) and (I_c) touch BC, CA, AB in S_a, T_a, U_a and S_b, T_b, U_b and S_c, T_c, U_c respectively. Prove that AS_a, BT_b, CU_c concur.

$\diamond$ The point in which AS_a, BT_b, CU_c concur is a remarkable point of ABC, called the *Nagel point* of ABC. It is usually denoted by N. $\diamond$

(c) Prove that N is the isotomic conjugate of Γ.

14. Introducing *isogonal conjugacy* with respect to a triangle ABC:

(a) Given a triangle ABC, consider $X \in BC - \{B, C\}$, $Y \in CA - \{C, A\}$, $Z \in AB - \{A, B\}$. Prove that

$$\frac{XB}{XC} \cdot \frac{YC}{YA} \cdot \frac{ZA}{ZB} = \frac{\sin(< XAB)}{\sin(< XAC)} \cdot \frac{\sin(< YBC)}{\sin(< YBA)} \cdot \frac{\sin(< ZCA)}{\sin(< ZCB)} \quad .$$

Derive from this relation " angular " versions of the theorems of Menelaus and Ceva.

(b) Consider lines k, l, m through the vertices A, B, C respectively. Let k', l', m' be reflections of k, l, m in the respective internal (or external) angle bisectors through A, B, C . Prove that k, l, m are concurrent or parallel iff k', l', m' are concurrent or parallel.

$\diamond$ If k, l, m concur in M (an ordinary point or a point "at infinity") and k', l', m' concur in $M^\star$ (again an ordinary point or a point "at infinity"), the points M and $M^\star$ are said to be *isogonal conjugates* of one another with respect to the triangle ABC. $\diamond$

(c) Prove that O and H are isogonally conjugate.

(d) Let M_a, M_b, M_c be the reflections of the point M in the sides BC, CA, AB. Prove that $AM^\star \perp M_bM_c$, $BM^\star \perp M_cM_a$, $CM^\star \perp M_aM_b$.

(e) Prove that $M^\star$ is the center of the circumcircle of the triangle $M_aM_bM_c$.

(f) Prove that $M^\star$ "lies at infinity" if and only if $M \in (O)$.

15. Introducing the *Lemoine point* of a triangle ABC:

$\diamond$ The isogonal conjugate of G is a remarkable point of ABC, called the *Lemoine point* (or sometimes the *Grebe point*) of ABC. It is usually denoted by K. $\diamond$

(a) Let K_1, K_2, K_3 be the feet of the perpendiculars from K onto BC, CA, AB. Prove that

$$|KK_1| : |KK_2| : |KK_3| = a : b : c.$$

(b) Prove and employ the identity

$$(x^2+y^2+z^2)(\alpha^2+\beta^2+\gamma^2) = (x\alpha+y\beta+z\gamma)^2+(y\gamma-z\beta)^2+(z\alpha-x\gamma)^2+(x\beta-y\alpha)^2$$

to show that the sum of the squares of distances of a point to the sides of ABC attains its minimum iff the point coincides with the Lemoine point of ABC.

(c) Let $BB''C'C$, $CC''A'A$, $AA''B'B$ be squares constructed "outside" on the respective sides BC, CA, AB of the triangle ABC. Let $C''A'$ and $A''B'$, $A''B'$ and $B''C'$, $B''C'$ and $C''A'$ intersect in $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ respectively. Prove that $A\tilde{A}$, $B\tilde{B}$, $C\tilde{C}$ concur in the Lemoine point of ABC. (*Hüseyin Demir*)

(d) Let u, v, w be the tangents to the circumcircle of ABC at A, B, C respectively. Let v and w, w and u, u and v intersect in K_a, K_b, K_c respectively. Prove that AK_a, BK_b, CK_c concur in the Lemoine point of ABC.