SELECTED PROBLEMS

I

1. In a triangle ABC, prove that

$$< BAH =< OAC.$$

2. In a triangle ABC, let AH, BH, CH meet BC, CA, AB in D, E, F respectively. Prove that

$$HA \cdot HD = HB \cdot HE = HC \cdot HF.$$

3. In the triangle ABC, let $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ be the feet of the altitudes through A, B, C respectively. Prove that H is either the incenter or one of the excenters of $\tilde{A}\tilde{B}\tilde{C}$. Which ? When ?

4. In a triangle ABC, prove that

$$a + b + c \geq m_a + m_b + m_c \geq \frac{3}{4}(a + b + c) \quad .$$

Can any of these inequalities degenerate into an equality ?

5. In the triangle ABC, let G_a, G_b, G_c be the centroids of HBC, HCA, HAB respectively. Prove that G is the orthocenter of $G_aG_bG_c$.

6. In a triangle ABC, let k, l, m be the perpendicular bisectors of $[B, C]$, $[C, A]$, $[A, B]$. Given a sequence X_n with $X_{3n} \in k$, $X_{3n+1} \in l$, $X_{3n+2} \in m$ such that $A \in X_{3n+1}X_{3n+2}$, $B \in X_{3n+2}X_{3n+3}$, $C \in X_{3n}X_{3n+1}$, prove that $X_n = X_{n+6}$ for all $n \in \mathbb{Z}$.

7. Prove that in a triangle ABC the following conditions are equivalent:

(a) $A = 90°$.
(b) $r + r_b + r_c = r_a$.
(c) $r_b r_c = r r_a$.

8. Evaluate A in a triangle ABC in which $3r + r_b + r_c = 3r_a$.

9. In a triangle ABC, prove that

$$\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} = \frac{1}{r} \quad .$$

Deduce that

$$h_a h_b h_c \geq 27r^3$$

equality holding iff the triangle is equilateral.

10. Introducing the *Bottema Triangle* :

In a triangle ABC, let the external angle bisectors through B and C intersect CA and AB in V and W respectively. Prove that $|BV| = |BC| = |CW|$ if and only if $A = 36°$, $B = 12°$, $C = 132°$.

⋄ The above introduced curious triangle is referred to as the *Bottema Triangle* after its instigator and constitutes a beautiful counterexample to a deceptively natural extension of the *Steiner-Lehmus Theorem.* ⋄

11. Prove that the circumcircle of a triangle passes through the midpoint of a line segment joining any two of the excenters.