Elementary Geometry, 2008 Fall

PROBLEMS (4)

**1.** (A) Given a triangle $UVW$, and a point $X$ on the circumcircle of $UVW$, prove that the Simson line of $X$ with respect to $UVW$ is parallel to $UY$ where $[XY]$ is the unique chord of the circumcircle of $UVW$ perpendicular to $VW$.

(B) Given a circle $\gamma$, consider points $P, C, D \in \gamma$ and let $[A, B]$ be a diameter of $\gamma$ . Prove that the Simson line of $P$ with respect to the triangle $ACD$ is perpendicular to the Simson line of $P$ with respect to the triangle $BCD$.

**2.** (AB) Let $A'$, $B'$, $C'$ be the respective midpoints of the line segments $[B, C]$, $[C, A]$, $[A, B]$. Prove that $O$ is the orthocenter of $A'B'C'$ .

(B) Consider $E$ which is the foot of the perpendicular from $B$ on $CA$. Let $U$, $W$ be the feet of perpendiculars from $E$ on $B'C'$, $A'B'$ respectively. Prove that $UW$ bisects $[B, E]$ .

(C) Prove that $UW$ is parallel to $OB$ .

**3.** Consider a triangle $ABC$ with circumcenter $O$ , circumcircle $(O)$. For each $X \in (O)$ let $l_X$ denote the Simson line of $X$ with respect to $ABC$.

(A) Prove that for any $X \in (O)$, $l_X$ is parallel to $AY$ where $[XY]$ is the unique chord of $(O)$ perpendicular to $BC$.

(B) Given $X \in (O)$, prove that in order for $l_X$ to be parallel to $OX$ it is necessary and sufficient that

$$3 \overset{\frown}{XA} = B - C \ .$$

(C) Prove that there exist exactly three points $X$ on $(O)$ such that $l_X$ is parallel to $OX$. Show that these points constitute an equilateral triangle.

(D) Prove that the Simson lines in question concur in the Euler point of $ABC$ . [1]

**4.** Given a triangle $ABC$, let the parallel to $BC$ through $A$ meet the circumcircle of $ABC$ in $L$ for a second time.

(A) Prove that the Simson line of $L$ is parallel to $OA$ where $O$ is the circumcenter of $ABC$.

(B) Let $[A, V]$ and $[L, U]$ be diameters of the circumcircle of $ABC$. Describe the Simson lines of $U$ and $V$.

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[1] Remember that $l_X$ bisects $[HX]$ .

**5.** In a triangle $ABC$ with orthocenter $H$ and circumcenter $O$ , let $A'$ denote the midpoint of $[B, C]$ .

(A) Prove that $|AH| = 2|A'O|$ .

(B) Prove that the circumcenter of $HBC$ is the reflection of $O$ in $BC$ .

(C) Consider distinct points $K$, $L$, $M$, $N$ on a circle $\Gamma$ . Let $T$, $U$, $V$, $W$ be the respective orthocenters of the triangles $LMN$, $KMN$, $KLN$, $KLM$. Prove that the line segments $[K, T]$, $[L, U]$, $[M, V]$, $[N, W]$ have a common midpoint .

(D) Prove that the respective Simson lines of $K$, $L$, $M$, $N$, with respect to $LMN$, $KMN$, $KLN$, $KLM$ are concurrent.

**6.** Let $ABC$ be a triangle with orthocenter $H$. Prove that a Simson line of $ABC$ is Simson line of $HBC$, $HCA$, $HAB$ as well.

**7.** Given a triangle $ABC$ and lines $\lambda, \mu$, prove that the following statements are equivalent:

(i) $\lambda, \mu$ are Simson lines of $ABC$ which are perpendicular to one another.

(ii) $\lambda, \mu$ are Simson lines of $ABC$ which are isotomic conjugates with respect to $ABC$.

(iii) $\lambda, \mu$ are isotomically conjugate with respect to $ABC$ and perpendicular to one another.

(iv) $\lambda, \mu$ are Simson lines of $ABC$ which concur on the nine-point circle of $ABC$.

**8.** In a triangle $ABC$ let $M \in (O)$, $k$ be the Simson line of $M$ with respect to $ABC$. Let $\alpha, \beta, \gamma, \delta \in k$ denote the respective feet of the the perpendiculars from $M$ onto $BC$, $CA$, $AB$, $k$.

(A) Prove that

$$|MB||MC| = 2R|M\alpha| \ .$$

(B) Prove that

$$|M\beta||M\gamma| = 2\rho|MA|$$

where $|M\delta| = 2\rho$.

(C) Prove that

$$|MA||M\alpha| = |MB||M\beta| = |MC||M\gamma| = 4R\rho \ .$$

(D) Prove that

$$|MA||MB||MC| = 8R^2\rho \ .$$

(E) Prove that

$$|M\alpha||M\beta||M\gamma| = 8R\rho^2 \ .$$