Elementary Geometry, 2008 Fall

PROBLEMS (3)

1. Given a triangle ABC, let A' be the midpoint of $[B, C]$ and consider $Y \in CA - \{C, A\}$, $Z \in AB - \{A, B\}$ such that BY and CZ meet on AA'. Prove that YZ is parallel to BC.

2. Prove that in a triangle ABC the altitude through A, the median through B and the internal angle bisector through C are concurrent iff $\sin A = \cos B \tan C$.

3. Given a triangle ABC, consider $P \in BC - \{B, C\}$, $Q \in CA - \{C, A\}$, $R \in AB - \{A, B\}$ such that AP, BQ, CR are concurrent. Let QR, RP, PQ meet BC, CA, AB in X, Y, Z respectively. Prove that

(A) X, Y, Z are collinear.

(B) AP, BY, CZ are concurrent or parallel.

4. Given a quadrilateral $ABCD$, consider $X \in AB - \{A, B\}$, $Y \in BC - \{B, C\}$, $Z \in CD - \{C, D\}$, $T \in DA - \{D, A\}$.

(A) Prove that if X, Y, Z, T are collinear then

$$\frac{XA}{XB} \cdot \frac{YB}{YC} \cdot \frac{ZC}{ZD} \cdot \frac{TD}{TA} = 1 \quad .$$

Is the converse true ?

(B) Given a triangle ABC, consider $L \in BC - \{B, C\}$, $M \in CA - \{C, A\}$, $N \in AB - \{A, B\}$ such that AL, BM, CN are concurrent. If X, Y, Z are midpoints of $[M, N]$ $[N, L]$, $[L, M]$ respectively prove that AX, BY, CZ are concurrent. [1]

5. Given a triangle ABC, let A', B', C' be midpoints of $[B, C]$, $[C, A]$, $[A, B]$ respectively. Consider $L \in BC - \{B, C\}$, $M \in CA - \{C, A\}$, $N \in AB - \{A, B\}$ such that AL, BM, CN are concurrent. If P, Q, R are midpoints of AL BM, CN, respectively prove that PA', QB', RC' are concurrent. [2]

6. Consider a quadrangle $ABCD$ with points P, Q, R, S, T on BD, AB, AD, CB, CD

[1] Let AX meet BC in U. Consider the quadrangle $MNBC$ to compute $\dfrac{UB}{UC}$.

[2] Observe that P, Q, R lie on the sides of the medial triangle $A'B'C'$.

respectively. Prove that if P, Q, R are collinear and P, S, T are collinear, then the lines QS, RT, AC are concurrent. [3]

7. The following is a problem by the late Prof. Demir who liked to present it with a quaintly arithmetic notation which is very suggestive and delightful. A numeral m written decimally will stand for a point and given such numerals m, n the line through them is denoted by $m \cdot n$:

Given distinct non-collinear points $1, 2, 3, 4$ consider points $12 \in 1 \cdot 2 - \{1, 2\}$, $23 \in 2 \cdot 3 - \{2, 3\}$, $34 \in 3 \cdot 4 - \{3, 4\}$. Let $1 \cdot 23$ and $12 \cdot 3$, $2 \cdot 34$ and $23 \cdot 4$ meet in 123, 234, respectively. Finally, let $1 \cdot 234$ intersect $123 \cdot 4$ in 1234. Prove that 12, 1234, 34 are collinear. [4]

8. Given triangle ABC, let D be a point on BC,and P, Q points on AB. Let PD meet AC in H, QD meet AC in K and CP meet AD in M and CQ meet AD in N. Prove that KM and HN meet on AB. [5]

9. Let ABC, DEF be triangles with AD, BE, CF concurrent. Prove that if AE, BF, CD are concurrent so are AF, BD and CE. [6]

10. Introducing *isotomic conjugacy* with respect to a triangle ABC:

(A) Consider $X \in BC - \{B, C\}$, $Y \in CA - \{C, A\}$, $Z \in AB - \{A, B\}$. Let $X' \in BC - \{B, C\}$, $Y' \in CA - \{C, A\}$, $Z' \in AB - \{A, B\}$. be reflections of X, Y, Z in the respective midpoints of $[B, C]$ $[C, A]$, $[A, B]$. Prove that AX, BY, CZ are concurrent or parallel iff AX', BY', CZ' are concurrent or parallel.

$\diamond$ If AX, BY, CZ concur in M (an ordinary point or a point "at infinity") and AX', BY', CZ' concur in M° (again an ordinary point or a point "at infinity"), the points M and M° are said to be *isotomic conjugates* of one another with respect to the triangle ABC. $\diamond$

(B) Prove that the areas of the triangles XYZ and $X'Y'Z'$ are equal.

(C) Prove that X', Y', Z' are collinear iff X, Y, Z are collinear.

$\diamond$ If X, Y, Z lie on the line m and X', Y', Z' lie on the line m° the lines m and m° are said to be *isotomic conjugates* of one another with respect to the triangle ABC.$\diamond$

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[3] The Desargues Theorem

[4] The Desargues Theorem.

[5] The dual of the Pappus Theorem.

[6] The dual of the Pappus Theorem.

11. Introducing the *Nagel and Gergonne points* of a triangle ABC:

(A) Let (I) touch BC, CA, AB in S, T, U respectively. Prove that AS, BT, CU concur.

⋄ The point in which AS, BT, CU concur is a remarkable point of ABC, called the *Gergonne point* of ABC. It is usually denoted by Γ. ⋄

(B) Let (I_a) and (I_b) and (I_c) touch BC, CA, AB in S_a, T_a, U_a and S_b, T_b, U_b and S_c, T_c, U_c respectively. Prove that AS_a, BT_b, CU_c concur.

⋄ The point in which AS_a, BT_b, CU_c concur is a remarkable point of ABC, called the *Nagel point* of ABC. It is usually denoted by N. ⋄

(C) Prove that N is the isotomic conjugate of Γ.

12. Introducing *isogonal conjugacy* with respect to a triangle ABC:

(A) Given a triangle ABC, consider $X \in BC - \{B, C\}$, $Y \in CA - \{C, A\}$, $Z \in AB - \{A, B\}$. Prove that

$$\frac{XB}{XC} \cdot \frac{YC}{YA} \cdot \frac{ZA}{ZB} = \frac{\sin(< XAB)}{\sin(< XAC)} \cdot \frac{\sin(< YBC)}{\sin(< YBA)} \cdot \frac{\sin(< ZCA)}{\sin(< ZCB)} \quad .$$

Derive from this relation " angular " versions of the theorems of Menelaus and Ceva.

(B) Consider lines k, l, m through the vertices A, B, C respectively. Let k', l', m' be reflections of k, l, m in the respective internal (or external) angle bisectors through A, B, C. Prove that k, l, m are concurrent or parallel iff k', l', m' are concurrent or parallel.

⋄ If k, l, m concur in M (an ordinary point or a point "at infinity") and k', l', m' concur in $M^\star$ (again an ordinary point or a point "at infinity"), the points M and $M^\star$ are said to be *isogonal conjugates* of one another with respect to the triangle ABC. ⋄

(C) Prove that O and H are isogonally conjugate.

(D) Let M_a, M_b, M_c be the reflections of the point M in the sides BC, CA, AB. Prove that $AM^\star \perp M_bM_c$, $BM^\star \perp M_cM_a$, $CM^\star \perp M_aM_b$.

(E) Prove that $M^\star$ is the center of the circumcircle of the triangle $M_aM_bM_c$.

(F) Prove that $M^\star$ "lies at infinity" if and only if $M \in (O)$.

13. Introducing the *Lemoine point* of a triangle ABC:

⋄ The isogonal conjugate of G is a remarkable point of ABC, called the *Lemoine point* (or sometimes the *Grebe point*) of ABC. It is usually denoted by K. ⋄

(A) Let K_1, K_2, K_3 be the feet of the perpendiculars from K onto BC, CA, AB. Prove that

$$|KK_1| : |KK_2| : |KK_3| = a : b : c.$$

(B) Prove and employ the identity

$$(x^2 + y^2 + z^2)(\alpha^2 + \beta^2 + \gamma^2) = (x\alpha + y\beta + z\gamma)^2 + (y\gamma - z\beta)^2 + (z\alpha - x\gamma)^2 + (x\beta - y\alpha)^2$$

to show that the sum of the squares of distances of a point to the sides of ABC attains its minimum iff the point coincides with the Lemoine point of ABC.

(C) Let $BB''C'C$, $CC''A'A$, $AA''B'B$ be squares constructed "outside" on the respective sides BC, CA, AB of the triangle ABC. Let $C''A'$ and $A''B'$, $A''B'$ and $B''C'$, $B''C'$ and $C''A'$ intersect in $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ respectively. Prove that $A\tilde{A}$, $B\tilde{B}$, $C\tilde{C}$ concur in the Lemoine point of ABC. [7]

(D) Let u, v, w be the tangents to the circumcircle of ABC at A, B, C respectively. Let v and w, w and u, u and v intersect in K_a, K_b, K_c respectively. Prove that AK_a, BK_b, CK_c concur in the Lemoine point of ABC.

[7] A result by Hüseyin Demir.