Elementary Geometry, 2008 Fall

PROBLEMS (2)

**1.** For a triangle $ABC$ with the standard notations, prove the following identities :

(A) $\Delta = \dfrac{abc}{4R}$ (B) $bc = 2Rh_a$ (C) $Rr = \dfrac{abc}{4s}$

(D) $\Delta = \dfrac{r_a r_b r_c}{s}$ (E) $\Delta = \sqrt{r r_a r_b r_c}$ (F) $\Delta = \sqrt{\dfrac{R h_a h_b h_c}{2}}$

(G) $bc + ca + ab = s^2 + r^2 + 4Rr$ [1] (H) $a^2 + b^2 + c^2 = 2s^2 - 2r^2 - 8Rr$

(I) $\left(h_a + h_b + h_c\right)\left(\dfrac{1}{h_a} + \dfrac{1}{h_b} + \dfrac{1}{h_c}\right) = \left(a + b + c\right)\left(\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c}\right)$

(J) $\dfrac{4R}{r} = \left(\dfrac{r_a}{r} - 1\right)\left(\dfrac{r_b}{r} - 1\right)\left(\dfrac{r_c}{r} - 1\right)$ (K) $r_b r_c + r_c r_a + r_a r_b = s^2$

(L) $\left(\dfrac{a}{r_a} + \dfrac{b}{r_b} + \dfrac{c}{r_c}\right)\left(\dfrac{a + b + c}{r_a + r_b + r_c}\right) = 4$ (M) $r_a + r_b + r_c = 4R + r$

(N) $\dfrac{1}{r^2} + \dfrac{1}{r_a^2} + \dfrac{1}{r_b^2} + \dfrac{1}{r_c^2} = \dfrac{a^2 + b^2 + c^2}{\Delta^2}$

(O) $\dfrac{1}{h_a} + \dfrac{1}{h_b} + \dfrac{1}{h_c} = \dfrac{1}{r}$ $(P)$ $\dfrac{1}{r_a} + \dfrac{1}{r_b} + \dfrac{1}{r_c} = \dfrac{1}{r}$

(Q) $a^2 + b^2 + c^2 + r^2 + r_a^2 + r_b^2 + r_c^2 = 16\Delta^2$ [2]

**2.** In a triangle $ABC$ prove that the following assertions are equivalent:

(i) $A = 90^\circ$ , (ii) $r + r_b + r_c = r_a$ , (iii) $r_b r_c = r r_a$ .

**3.** Compute the angle $A$ in a triangle $ABC$ in which

$$3r + r_b + r_c = 3r_a \ .$$

**4.** In a triangle $ABC$ prove that

$$m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}\left(a^2 + b^2 + c^2\right) \ .$$

[1] Square the Heron formula, employ (A) .

[2] Combine (L), (J) and (G) .

**5.** In a triangle $ABC$ prove that

$$\Delta = \frac{4}{3}\sqrt{\mu(\mu - m_a)(\mu - m_b)(\mu - m_c)}$$

where $2\mu = m_a + m_b + m_c$ . [3]

**6.** For a positively oriented triangle $ABC$, prove the following identities :

(A) $\sin A + \sin B + \sin C = 4\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right)$ [4]

(B) $\sin A + \sin B - \sin C = 4\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right)$

(C) $\cos A + \cos B + \cos C = 1 + 4\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)$

(D) $\sin 2A + \sin 2B + \sin 2C = 4\sin A\sin B\sin C$

(E) $\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A\cos B\cos C$

(F) $\sin^2 A + \sin^2 B + \sin^2 C = 2\Big(1 + \cos A\cos B\cos C\Big)$

(G) $\tan A + \tan B + \tan C = \tan A\tan B\tan C$

(H) $\tan\left(\frac{A}{2}\right) = \frac{r}{s-a}$ [5]

(I) $\tan\left(\frac{A}{2}\right) = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$ (J) $\sin\left(\frac{A}{2}\right) = \sqrt{\frac{(s-b)(s-c)}{bc}}$

(K) $\cos\left(\frac{A}{2}\right) = \sqrt{\frac{s(s-a)}{bc}}$ (L) $\sin A + \sin B + \sin C = \frac{s}{R}$

(M) $\sin B\sin C + \sin C\sin A + \sin A\sin B = \frac{s^2 + r^2 + 4R^2}{R^2}$

(N) $\cos B\sin C + \cos C\sin A + \cos A\sin B = \frac{s^2 + r^2 - 4R^2}{R^2}$

(O) $\cos A + \cos B + \cos C = 1 + \frac{r}{R}$ (P) $\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right) = \frac{r}{4R}$

(Q) $\tan\left(\frac{A}{2}\right)\tan\left(\frac{B}{2}\right)\tan\left(\frac{C}{2}\right) = \frac{r}{s}$ (R) $\frac{\sin(A-B)}{\sin(A+B)} = \frac{a^2-b^2}{c^2}$

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[3]Consider the triangle $AA'K$ presented in the Problems (1) Pr. 5 . Compare its area with the area of $ABC$.

[4]Notice that $B = (B+C)/2 + (B-C)/2$ , $C = (B+C)/2 - (B-C)/2$ , $A = \pi/2 - (B+C)/2$ . Start with the second and the third term on the left hand side .

[5]Let $U$ be the point in which the incircle touches $AB$. Consider the triangle $AUI$ .

**7.** In a triangle $ABC$ with area $\Delta$ prove that

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}\Delta \quad . \text{[6]}$$

Deduce that

$$a \sin A + b \sin B + c \sin C \geq \frac{2\sqrt{3}\Delta}{R}$$

equality holding iff the triangle is equilateral.

**8.** In a positively oriented triangle $ABC$, prove the following identities :

(A) $a^2 + b^2 + c^2 = 8R^2(1 + \cos A \cos B \cos C)$

(B) $a \sin A + b \sin B + c \sin C = \dfrac{a^2 + b^2 + c^2}{2R}$

(C) $a \sin A + b \sin B + c \sin C = \dfrac{h_b h_c}{h_a} + \dfrac{h_c h_a}{h_b} + \dfrac{h_a h_b}{h_c} \quad .$

**9.** Prove that

(A) $2s^2 \geq 27Rr$ (B) $h_a h_b h_c \geq 27r^3$

in both cases equality holding iff the triangle is equilateral. [7]

**10.** In a positively oriented triangle $ABC$, prove that

$$9r \leq a \sin A + b \sin B + c \sin C \leq \frac{9}{2}R$$

in particular $2r \leq R$, equality holding iff the triangle is equilateral. [8]

**11.** In a positively oriented triangle $ABC$, prove that

$$\sin A + \sin B + \sin C = \frac{2R}{r} \sin A \sin B \sin C \leq \frac{3\sqrt{3}}{4}\frac{R}{r} \leq \frac{3\sqrt{3}}{2}$$

equality holding iff the triangle is equilateral. [9]

**12.** In a triangle $ABC$, let $M$ be a point on $BC$ such that $MB : MC = -1 : 2$ . Employ the Steward relation (or not) to prove that

$$|AM|^2 = \frac{1}{3}b^2 + \frac{2}{3}c^2 - \frac{2}{9}a^2 \quad .$$

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[6] $a^2 + b^2 + c^2 - 4\sqrt{3}\Delta = 2(b^2 + c^2) - 2bc \cos A - 4\sqrt{3}\Delta$ .

[7] Remember that the arithmetic mean exceeds the geometric mean. Employ 1(A), 1(O) .

[8] Remember that the arithmetic mean exceeds the geometric mean. Employ 8(A), 8(C), 1(O) .

[9] The left hand side equals $s/R = \Delta/Rr$ . Employ 1(A) .

**13.** In a triangle $ABC$, let $A'$ be the midpoint of $[BC]$. Prove the following identities :

(A) $|OA'|^2 = R^2 - \dfrac{a^2}{4}$

(B) $|OG|^2 = R^2 - \dfrac{a^2+b^2+c^2}{9}$ .

**14.** In a triangle $ABC$, let $A'$ be the midpoint of $[BC]$, let $S$ be the point in which the incircle touches $BC$ . Prove the following identities :

(A) $|SA'| = \dfrac{b-c}{2}$

(B) $|A'I|^2 = r^2 + \left(\dfrac{b-c}{2}\right)^2$

(C) $|AI|^2 = \dfrac{4bc(s-a)}{s}$

(D) $|GI|^2 = \dfrac{s^2+5r^2-16Rr}{9}$ .

**15.** Consider distinct fixed points $A, B, C, D$ and a variable point $P$ on a given circle $\Gamma$. Let $PB, PC$ intersect $AD$ in $X, Y$ respectively and put $x = |AX|$, $y = |DY|$, $z = |XY|$. Prove the *Haruki Lemma* to the effect that the quantity $xyz^{-1}$ is independent of the position of $P$ on $\Gamma$. [10]

[10] Compute and compare the areas of the triangles $PAX$, $PYD$, $PXY$.