Introduction to Algebra

MATH 370

SECOND MIDTERM

(*Duration : 90 mins.*)

11th May 1996

[10 + 10 + 10] , [10 + 10 + 10 + 10] , [10 + 10 + 10]

1.

Consider $\mathbb{Z}_n$, 'the ring of integers modulo n' . A typical element of $\mathbb{Z}_n$ will be denoted by $[x]_n$, standing for the residue class modulo n containing $x \in \mathbb{Z}$. Prove that there exists a well-defined homomorphism $\varphi : \mathbb{Z}_4 \longrightarrow \mathbb{Z}_{10}$ defined by $\varphi([x]_4) = [5x]_{10}$. What is the kernel of this homomorphism ?

2.

True or false ? Prove briefly if true, provide a counterexample if not :
(a) If $o(G) = n$, then $o(\mathrm{Aut}(G)) = n!$.
(b) If G is abelian then so is $\mathrm{Aut}(G)$.
(c) If G is a finite simple group, then each non-trivial homomorphism of G into G is an automorphism.
(d) There exists a non-commutative ring R with the property that $x^2 = x$ for all $x \in R$.

3.

Let R be a commutative ring with multiplicative identity. Define $\sqrt{R} \subseteq R$ by

$$\sqrt{R} = \{ \ x \in R \mid x^n = 0 \text{ for some } n \geq 1 \ \}$$

(a) Show that $\sqrt{R}$ is an ideal of R .
(b) Prove that $\sqrt{R} \subseteq \ker\varphi$ for any homomorphism $\varphi : R \longrightarrow S$, where S is an integral domain.
(c) Determine $\sqrt{R/\sqrt{R}}$.

✓

Introduction to Algebra

MATH 370

SECOND MIDTERM

(Duration : 90 mins.)

11th May 1996

[10 + 10 + 10] , [10 + 10 + 10 + 10] , [10 + 10 + 10]

1.

Consider $\mathbb{Z}_n$, 'the ring of integers modulo n' . A typical element of $\mathbb{Z}_n$ will be denoted by $[x]_n$, standing for the residue class modulo n containing $x \in \mathbb{Z}$. Prove that there exists a well-defined homomorphism $\varphi : \mathbb{Z}_4 \longrightarrow \mathbb{Z}_{10}$ defined by $\varphi([x]_4) = [5x]_{10}$. What is the kernel of this homomorphism ?

a) Well definedness: $[x]_4 = [x']_4 \Longrightarrow 4 \mid x - x' \Longrightarrow 10 \mid 5x - 5x'$

$$\Longrightarrow [5x]_{10} = [5x']_{10}$$

b) Homomorphism : $\varphi([x]_4 + [y]_4) = \varphi([x+y]_4) = [5(x+y)]_{10} = [5x]_{10} + [5y]_{10}$

$$= \varphi([x]_4) + \varphi([y]_4)$$

$$\varphi([x]_4 [y]_4) = \varphi([xy]_4) = [5xy]_{40} = [25xy] = [5x]_{10}[5y]_{10}$$

$$= \varphi([x]_4)\varphi([y]_4).$$

c) $\ker \varphi = \{ [x]_4 \in \mathbb{Z}_4 \mid \varphi([x]_4) = [5x]_{10} = [0]_{10} \}$

$$= \{ [x]_4 \in \mathbb{Z}_4 \mid 5x \equiv 0 \pmod{10} \}$$

$$= \{ [x]_4 \in \mathbb{Z}_4 \mid x \equiv 0 \pmod 2 \} = \{ [0]_4 , [2]_4 \}$$

2.

True or false ? Prove briefly if true, provide a counterexample if not :

(a) If $o(G) = n$, then $o(\mathrm{Aut}(G)) = n!$.
(b) If G is abelian then so is $\mathrm{Aut}(G)$.
(c) If G is a finite simple group, then each non-trivial homomorphism of G into G is an automorphism.
(d) There exists a non-commutative ring R with the property that $x^2 = x$ for all $x \in R$.

(a) FALSE : $\mathrm{Aut}(\mathbb{Z}_{17}) \simeq \mathbb{Z}_{16}$

(b) FALSE : $\mathrm{Aut}(\mathbb{R}^2) \simeq GL(\mathbb{R}, 2)$ which is non-abelian.

(c) TRUE : $\ker\varphi \trianglelefteq G$; but $\ker\varphi \neq G$ as φ is non-trivial. Hence $\ker\varphi = \{e\}$ and φ is injective. G being finite, φ is an automorphism.

(d) FALSE : A ring R with the property that $x^2 = x$ for all $x \in R$, is necessarily commutative. From $(2x)^2 = x^2 + 2x + x^2 = x + x$ and $x^2 = x$ we may deduce $2x = 0$ or $-x = x$. From $(x+y)^2 = x+y$ and $x^2 = x$, $y^2 = y$ it is seen that $xy = yx$ for any $x, y \in R$.

3.

Let R be a commutative ring with multiplicative identity. Define $\sqrt{R} \subseteq R$ by

$$\sqrt{R} = \{ x \in R \mid x^n = 0 \text{ for some } n \geq 1 \}$$

(a) Show that $\sqrt{R}$ is an ideal of R .
(b) Prove that $\sqrt{R} \subseteq \ker\varphi$ for any homomorphism $\varphi : R \longrightarrow S$, where S is an integral domain.
(c) Determine $\sqrt{R/\sqrt{R}}$.

(a) $0 \in \sqrt{R}$ obviously. If $x, y \in \sqrt{R}$, there exist $m, n \geqslant 0$ such that $x^m = y^n = 0$. Consequently $x \pm y$ $(x+y)^{m+n} = \sum_{\substack{p+q=m+n \\ p,q \geqslant 0}} x \binom{m+n}{p} x^p y^q = 0$

and $x+y \in \sqrt{R}$. Similarly $x-y$. (Note the role of commutativity of R in the validity of Pascal's formula ; moreover $p \geqslant m$ or $q \geqslant n$ if $p+q = m+n$)